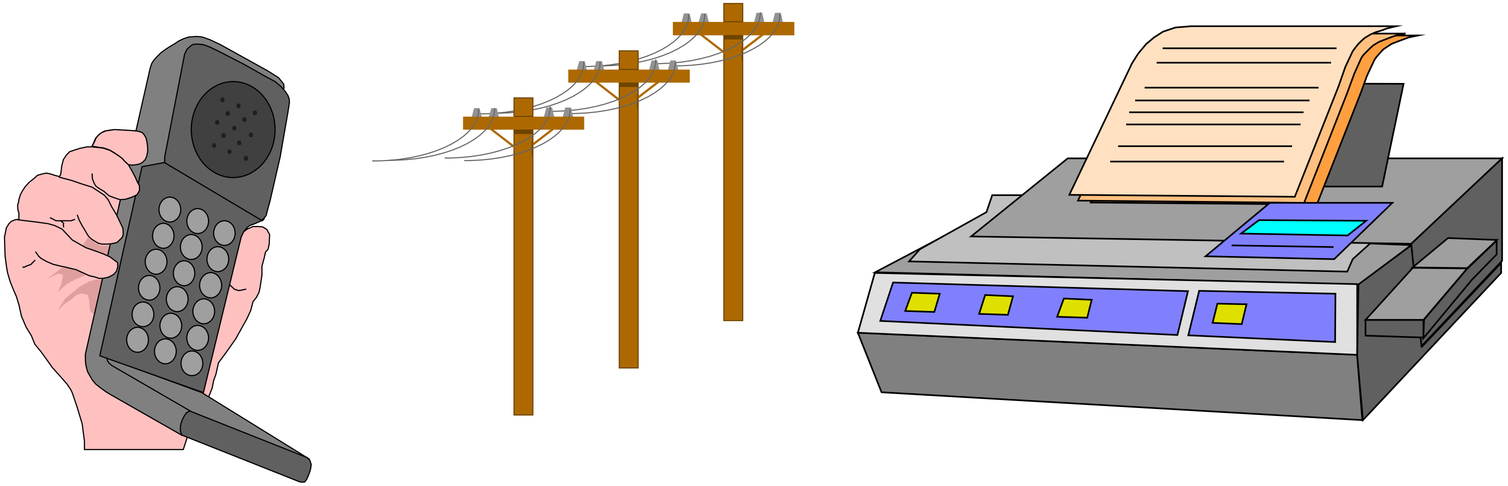
Digital Communications



An Introduction to the Mathematics of Sampled Signals

W.G.Marshall

Recommended Book

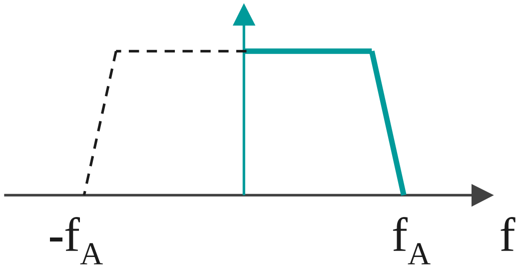
Introductory Digital Signal Processing

Authors: Lynn & Fuerst

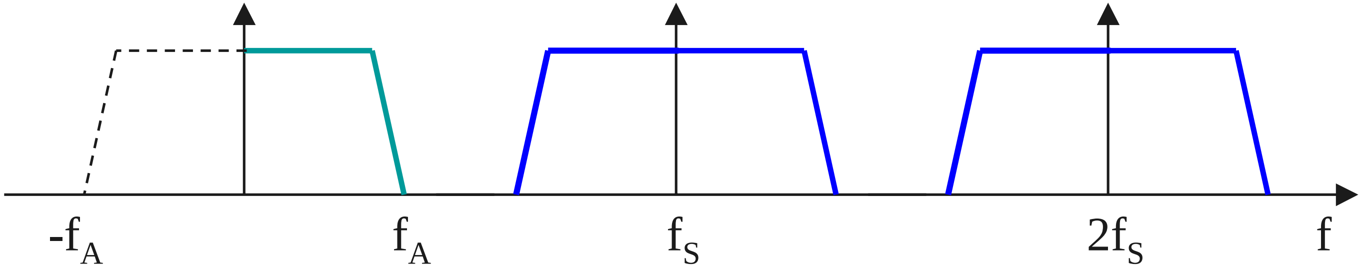
Publisher: Wiley

● Sampled Signals

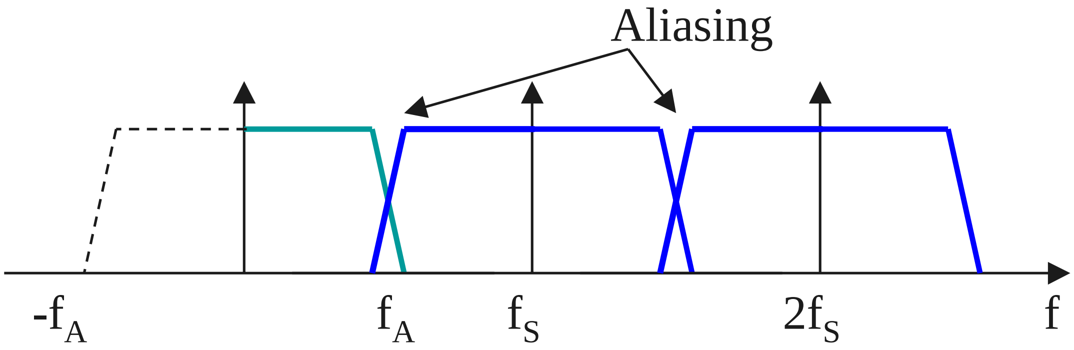
Original Spectrum (Baseband)



Sampled Spectrum $f_S >= 2f_A$

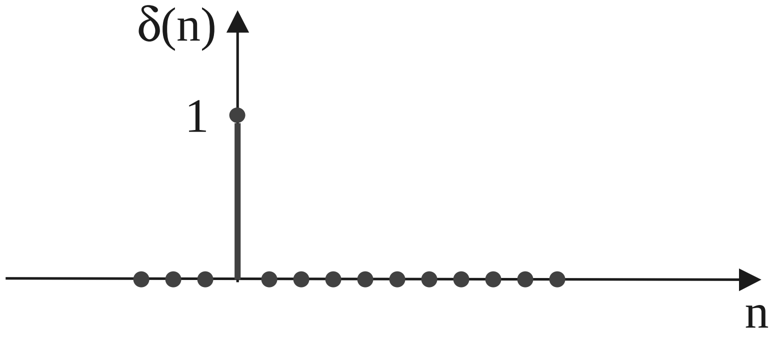


Sampled Spectrum $f_S < 2f_A$



- **Basic Sampled Signals**

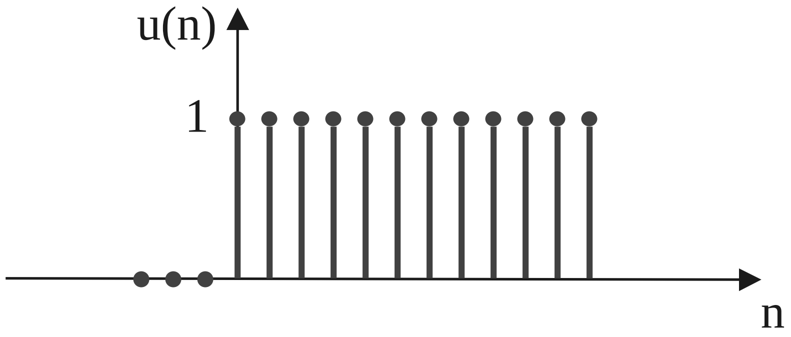
Unit Sample or Impulse Function $\delta(n)$



$\delta(n) = 0$ for $n \neq 0$

$\delta(n) = 1$ for $n = 0$

Unit Step Function



$u(n) = 0$ for $n < 0$

$u(n) = 1$ for $n \geq 0$

u(n) is the Running Sum of δ(n):

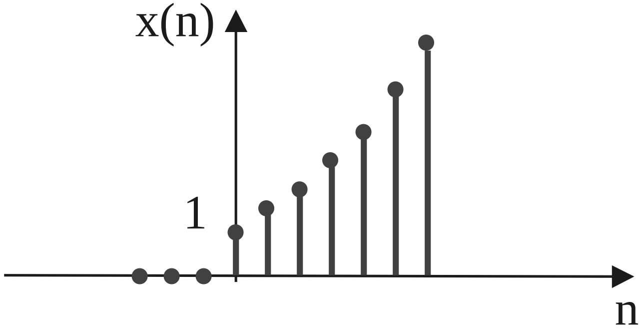
$$u(n) = \sum_{m=-\infty}^{n} (m)$$

Conversely, we can derive δ(n) from u(n):

$$\delta(n) = u(n) - u(n-1)$$

Recurrence formula holds for all integer n.

Exponential Function



$x(n) = a^n$ for $n \geq 0$

$x(n) = 0$ for $n < 0$

<u>or</u> $x(n) = a^n u(n)$

‘a’ is real, but can be imaginary or complex

Decaying for |a| < 1, Increasing for |a| > 1

Discrete Time Systems

Input Sequence → System → Output Sequence

(Excitation) (Response)

$x(n)$ $y(n)$

$$y(n) = H[x(n)]$$

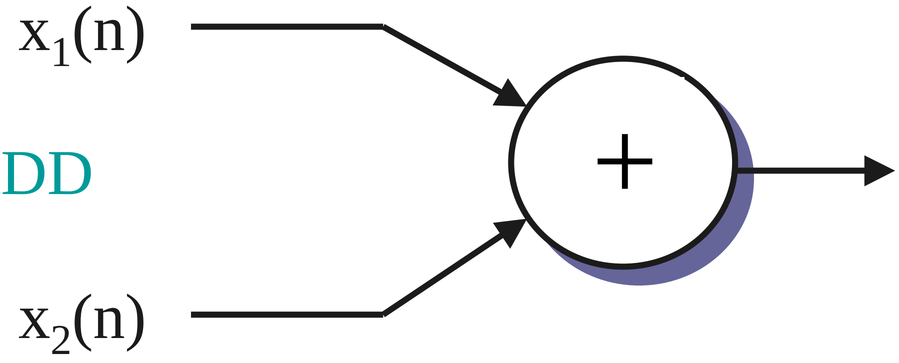
Examples

Shift by 1 sample (delay): $y(n) = x(n-1)$

Average: $y(n) = 0.5(x(n) + x(n-1))$

Basic Signal Flow Graphs

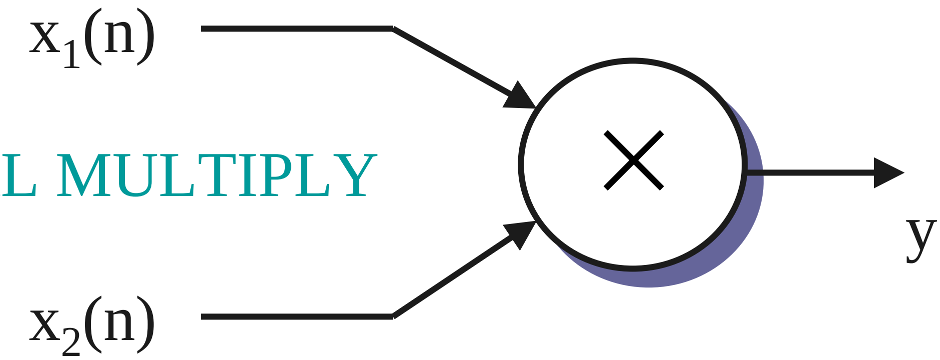
ADD



$y(n) = x_1(n) + x_2(n)$

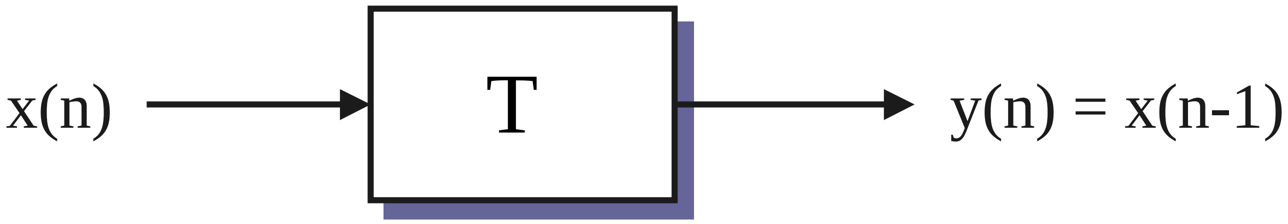
CONSTANT MULTIPLY $x(n) \xrightarrow{\quad a \quad} y(n) = a\, x(n)$

SIGNAL MULTIPLY



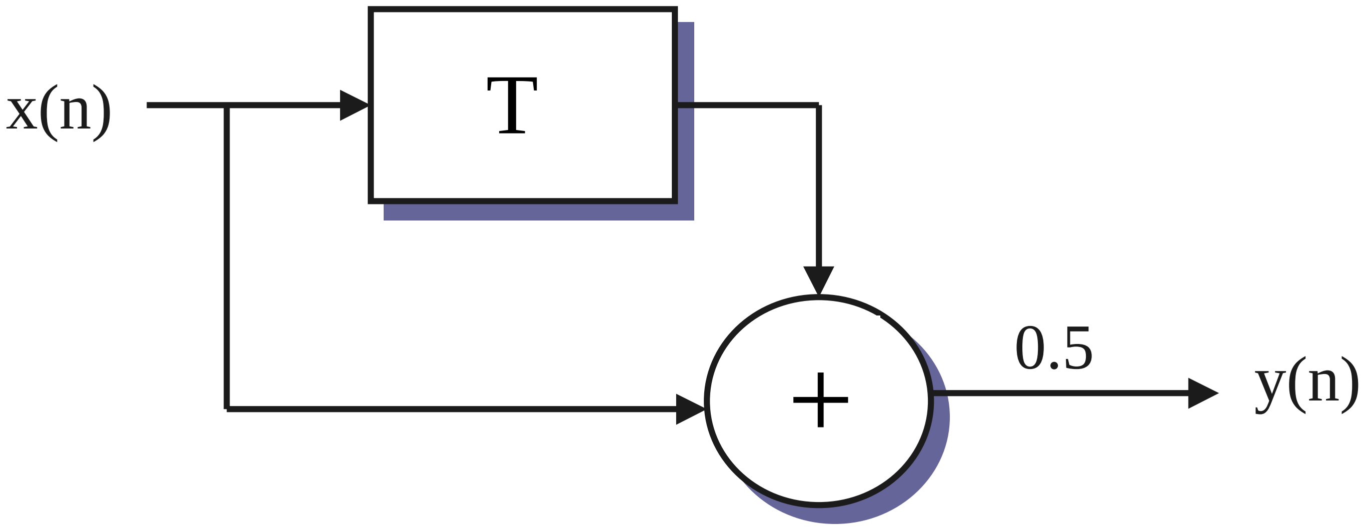
$y(n) = x_1(n)\, x_2(n)$

UNIT DELAY



AVERAGE

y(n) = 0.5(x(n) + x(n-1))



- **Linear Time-Invariant (LTI) Systems**

  Linear

  implies Superposition:

  $$H[a\ x(n)] = a\ H[x(n)]$$

  and $H[x_1(n) + x_2(n)] = H[x_1(n)] + H[x_2(n)]$

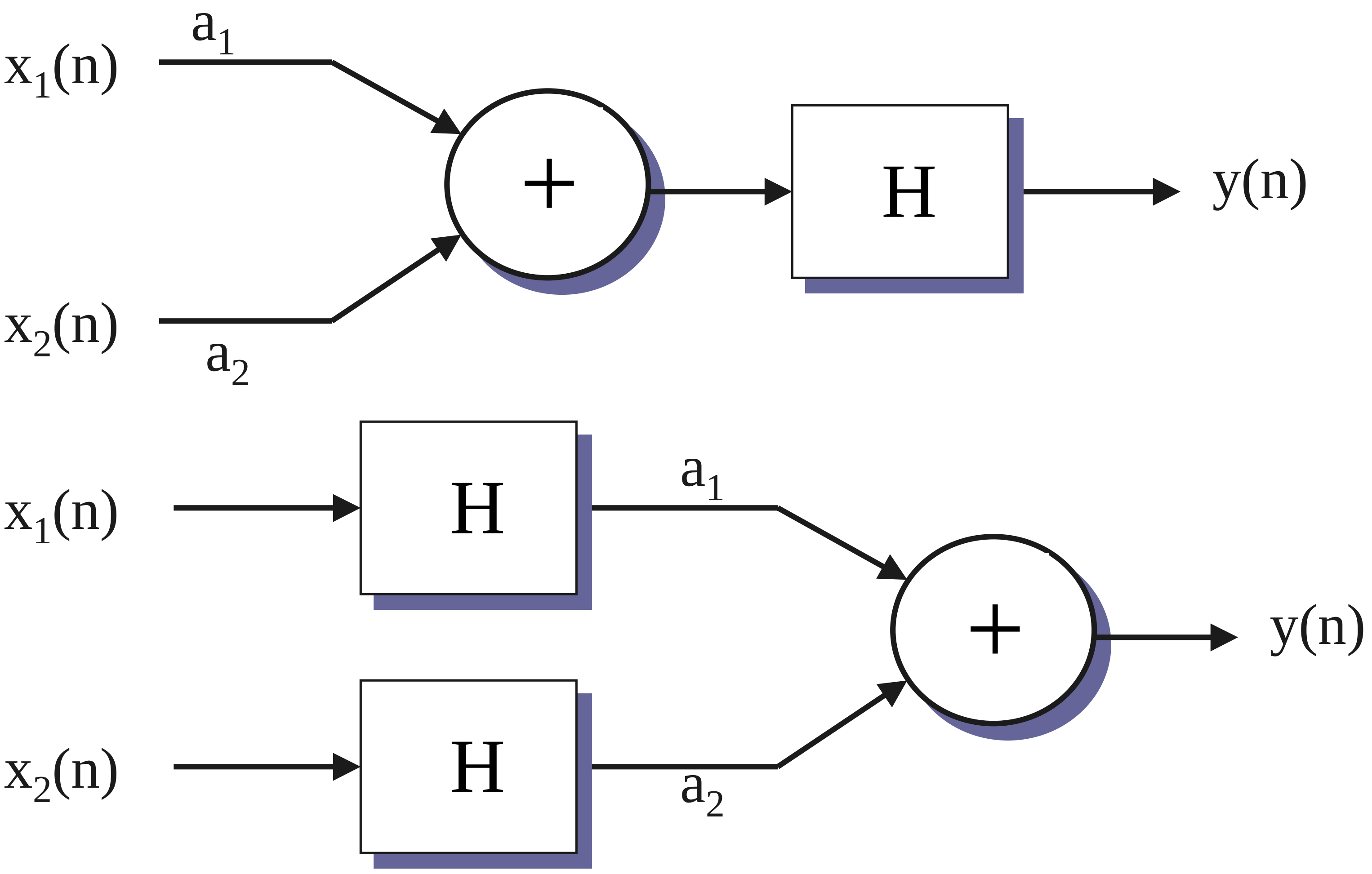
  where a = constant

  i.e. Scaling and addition translate throughout the function.

  These properties combine as:

$$H[a_1x_1(n) + a_2x_2(n)] = a_1H[x_1(n)] + a_2H[x_2(n)]$$

If H is linear, then the following two circuits have identical outputs:



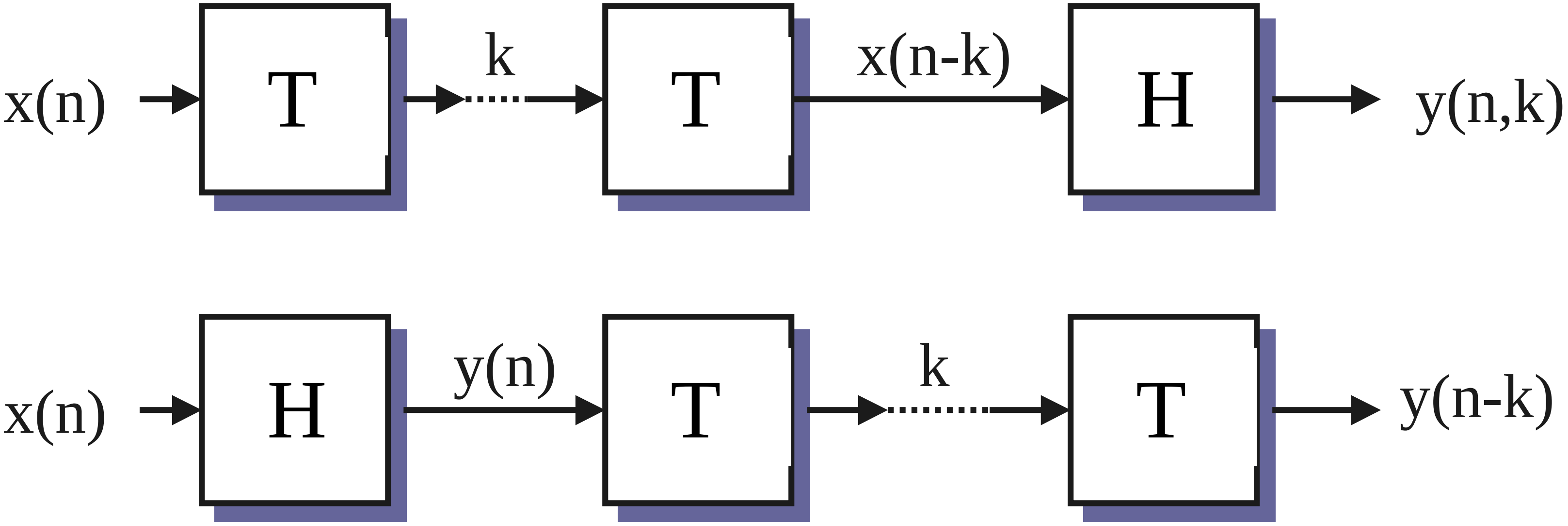
Time-Invariant

define y(n,k) = H[x(n-k)]

H is time-invariant if:

y(n,k) = y(n-k) for all k

i.e. the following two circuits have identical outputs:



- **Causality**

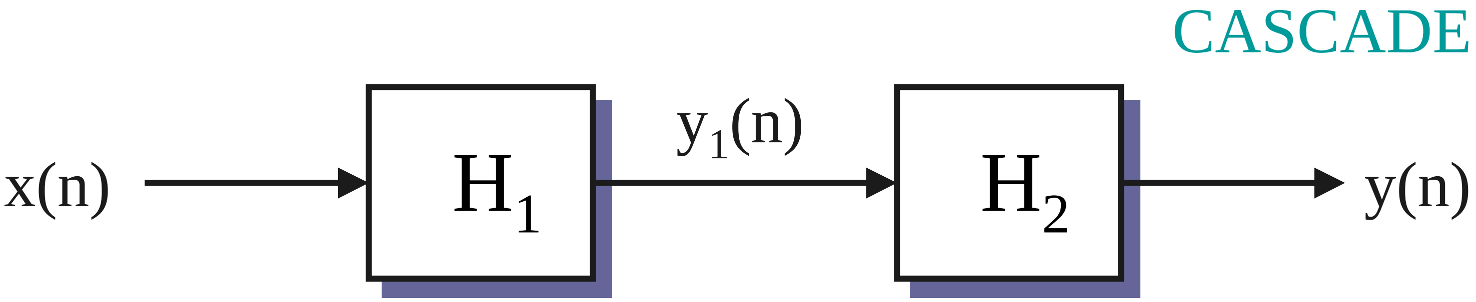
A system is said to be *Causal* if the present output depends only on the past and present inputs:

y(n) = H[x(n)] = f(x(n), x(n-1), x(n-2)……)

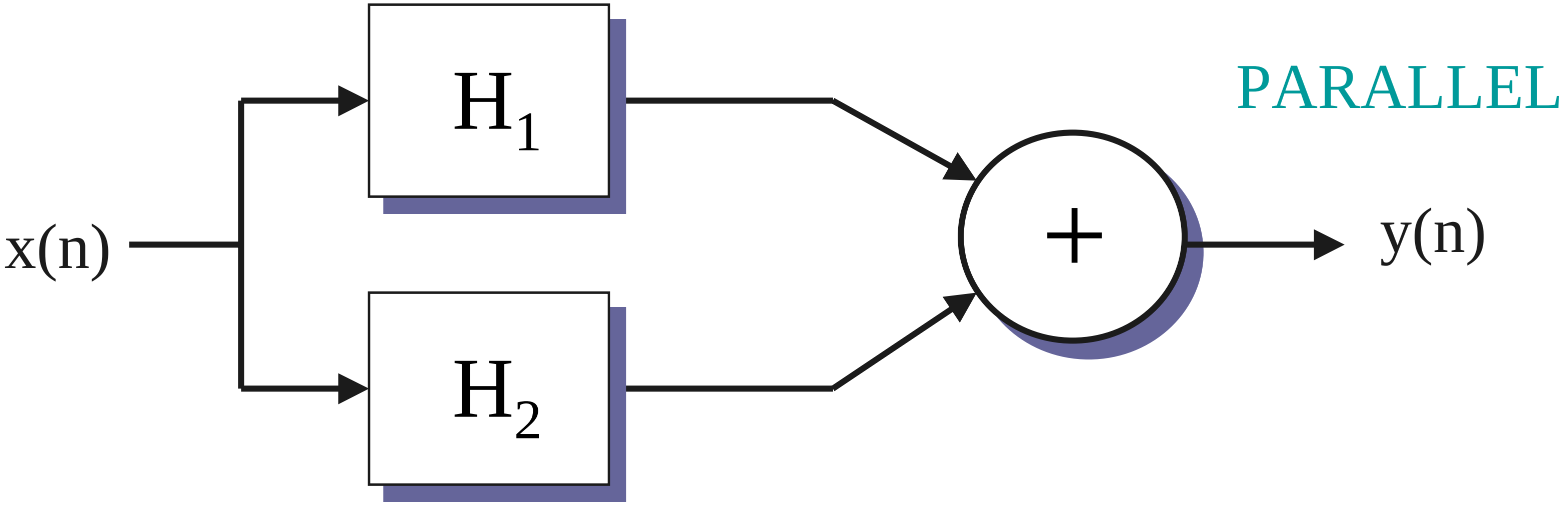
That is, the terms x(n+1), x(n+2)…. do not appear in the expression for H[x(n)].

In real-time systems, only causal relationships are possible. Non-causal systems are only possible if inputs are pre-stored and operated on later. This puts a *lag* into the system.

● System Interconnection



$$y_1(n) = H_1[x(n)] \quad y(n) = H_2[y_1(n)] = H_2[H_1[x(n)]] = H_2H_1[x(n)]$$



$$y(n) = H_1[x(n)] + H_2[x(n)] = (H_1 + H_2)[x(n)]$$

- **Closed form of Geometric Series**

  $A = \sum a^n$ for $n = 0$ to ∞ (Infinite length)

  for the case $|a| < 1$ the series converges.

  $A = 1 + a + a^2 + a^3 + a^4 + \ldots\ldots.$

  $A - 1 = a + a^2 + a^3 + a^4 + \ldots\ldots.$

  $= a(1 + a + a^2 + a^3 + a^4 + \ldots\ldots.)$

  $= aA$

  Hence $A = 1/(1 - a)$

  This is a lot easier to handle than the open form.

- **Exponentials, Sines and Cosines**

The ‘imaginary’ exponential sequence is defined:

$x(n) = e^{j\Omega n}$ Ω = constant

from Euler: $e^{j\Omega n} = \cos(\Omega n) + j\sin(\Omega n)$

With further manipulation it can be shown that:

$\cos(\Omega n) = \frac{1}{2}e^{j\Omega n} + \frac{1}{2}e^{-j\Omega n}$ Similarly for sine

Remember: n is just a dimensionless integer variable so Ω = constant, range $0 \rightarrow 2\pi$ rads.

cf. $x(t) = \sin(\omega t)$ for continuous-time system

ω = frequency in rads/sec because t has dimension of time (seconds).

- **Periodicity**

x(n) periodic if using integer N:

$x(n) = x(n+N)$ where N is the <u>Period</u>

otherwise the signal is Aperiodic.

Example: $x(n) = \sin(\Omega n)$ $-\infty < n < \infty$

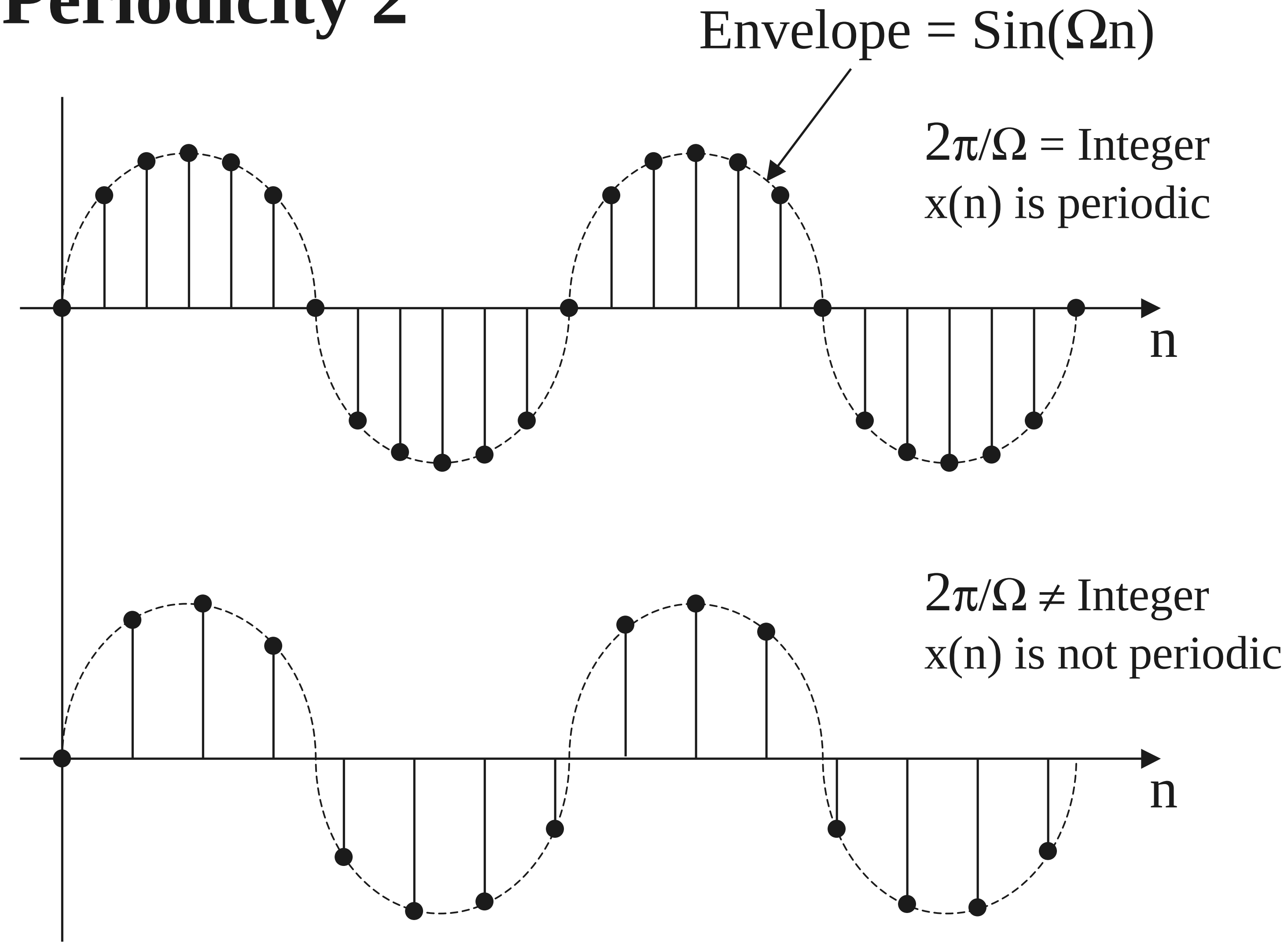
$\sin[\Omega(n+N)] = \sin(\Omega n)$ to be periodic

i.e. Consecutive sinusoids may only differ by integer multiples of 2π. Hence: $\Omega N = 2\pi$

Period $N = 2\pi/\Omega$

The ‘envelope’ of the sampled sinusoid is periodic, but the sampled values do not have to be.

- **Periodicity 2**



- **Sampled versus Continuous-Time**

  ω is frequency, rads/sec. Ω is angle, rads.

  Define the no. of samples in each sinusoid (single envelope cycle).

  One cycle period $n\Omega = 2\pi$

  i.e. $n = 2\pi/\Omega$ samples per period *regardless of time or frequency scales of the original*.

  Example: Sinusoid has 10 samples/period

  $N = 2\pi/\Omega = 10 \quad \Omega = \pi/5$

  Hence signal $x(n) = \sin(n\pi/5)$

- **Converting from ‘n’ to ‘t’**

  Often need to work in terms of time and frequency.

` Use the Sample Period T, hence sampling instants given by:

$t = nT$ $n = \ldots -2, -1, 0, 1, 2, 3 \ldots\ldots$

Sampling Frequency $f_S = 1/T$

Digital signal becomes:

$x(n) = \sin(\Omega n) = \sin(\omega nT)$ nT = time in secs

As $\omega = 2\pi f$ then $x(n) = \sin(n2\pi f/ f_S)$

- **Complex Exponentials**

  Consider $x(n) = e^{(a + j\Omega)n} = e^{an} e^{j\Omega n}$

  the ‘frequency’ term is modulated by the ‘amplitude ‘ term leading to:

  $x(n) = e^{an} \sin(n\Omega)$ or $x(n) = e^{an} \cos(n\Omega)$

  This is a sinusoidal envelope either rising or decaying exponentially:

  $a < 0$ decaying

  $a = 0$ no increase or decay

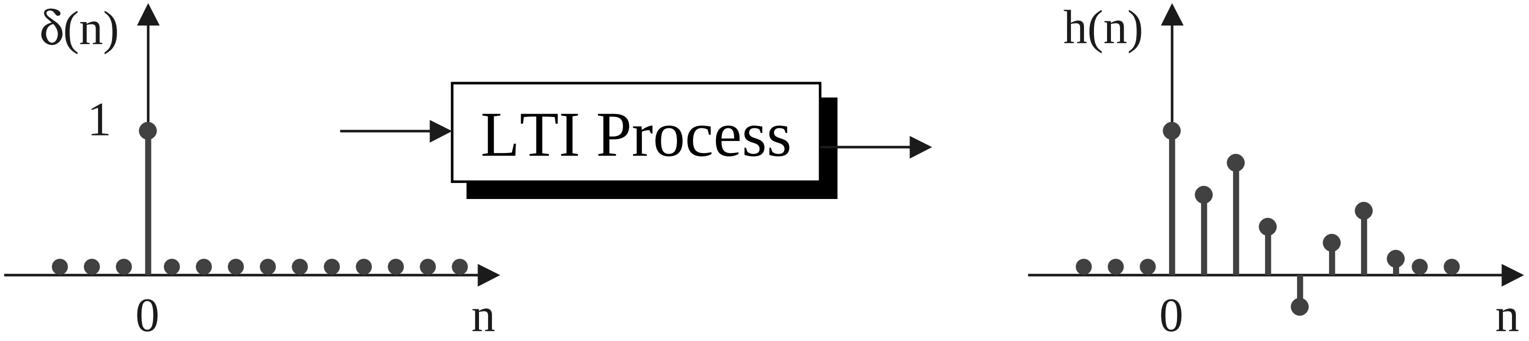
  $a > 0$ increasing (unstable system!)

- **Impulse Response h(n)**

  Defines behaviour of digital LTI process in the Time Domain. c.f. Frequency Response

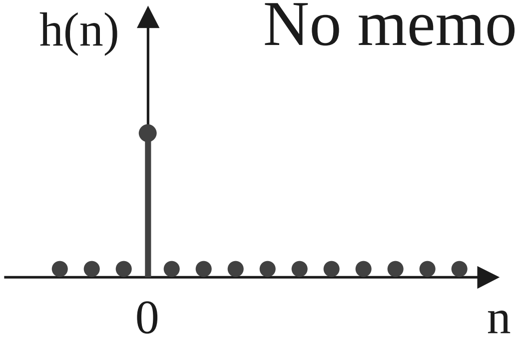
  Also known as the *Natural Response*.

  h(n) is the output from an LTI system when the input is δ(n).   h(n) = H [δ(n)]

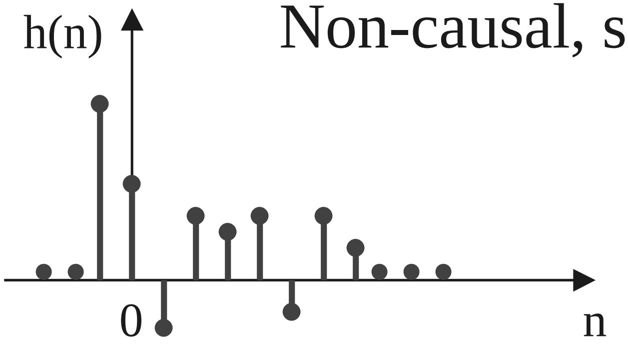


Examples of Impulse Response

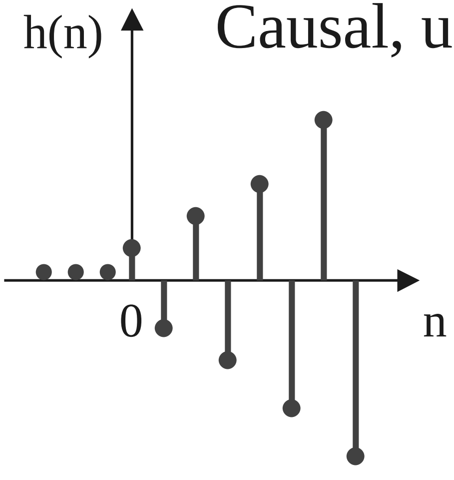
No memory



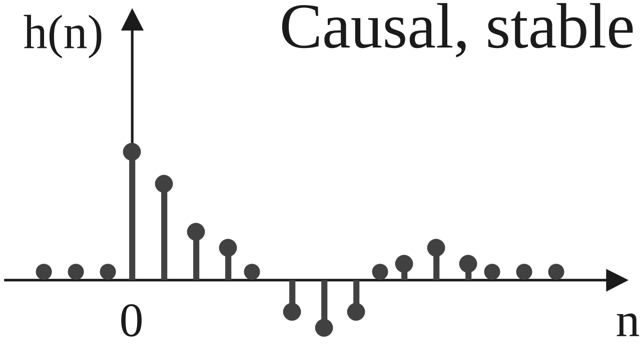
Non-causal, stable



Causal, unstable



Causal, stable



- **Example: Find the System Impulse Response**

Find the first four sample values of h(n):

$$y(n) = -0.8\, y(n-1) + x(n)$$

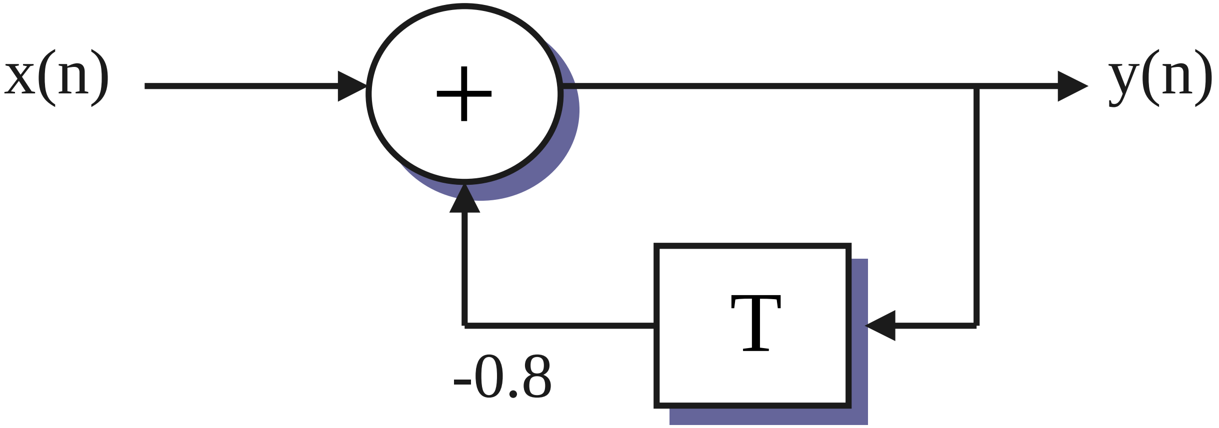
Hence $h(n) = -0.8\, h(n-1) + \delta(n)$

System is Causal, so $h(n) = 0$ for $n < 0$

$h(0) = -0.8\, h(-1) + \delta(0) \quad = \quad 0 + 1 = 1$

$h(1) = -0.8\, h(0) = -0.8$ $\qquad$ $h(2) = -0.8\, h(1) = 0.64$

$h(3) = -0.8\, h(2) = -0.512$



- **Step Response s(n)**

  ‘Step’ waveforms are very common in ‘real’ systems. e.g. Pulse train is a series of positive and negative steps.

  Step response is the running sum of the impulse response:

$$s(n) = \sum_{m=-\infty}^{n} h(m)$$

or h(n) is the first-order difference of s(n):

$h(n) = s(n) - s(n-1)$

s(n) provides the same information as h(n)

- **Example: Finding the System Step Response**

Find the first four sample values of h(n):

$$y(n) = 0.7\, y(n-1) + x(n)$$

Hence $h(n) = 0.7\, h(n-1) + \delta(n)$

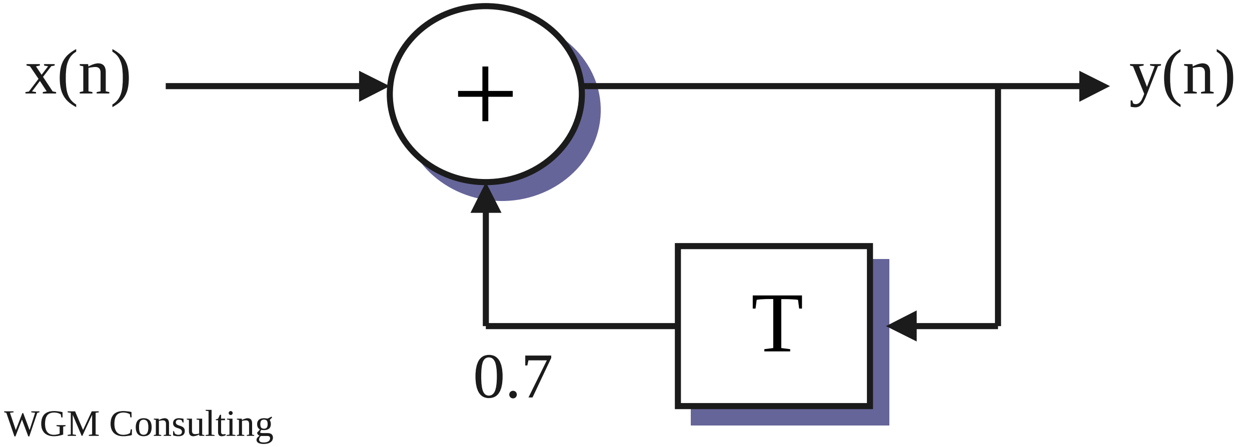
$h(0) = 1$ $h(1) = 0.7$

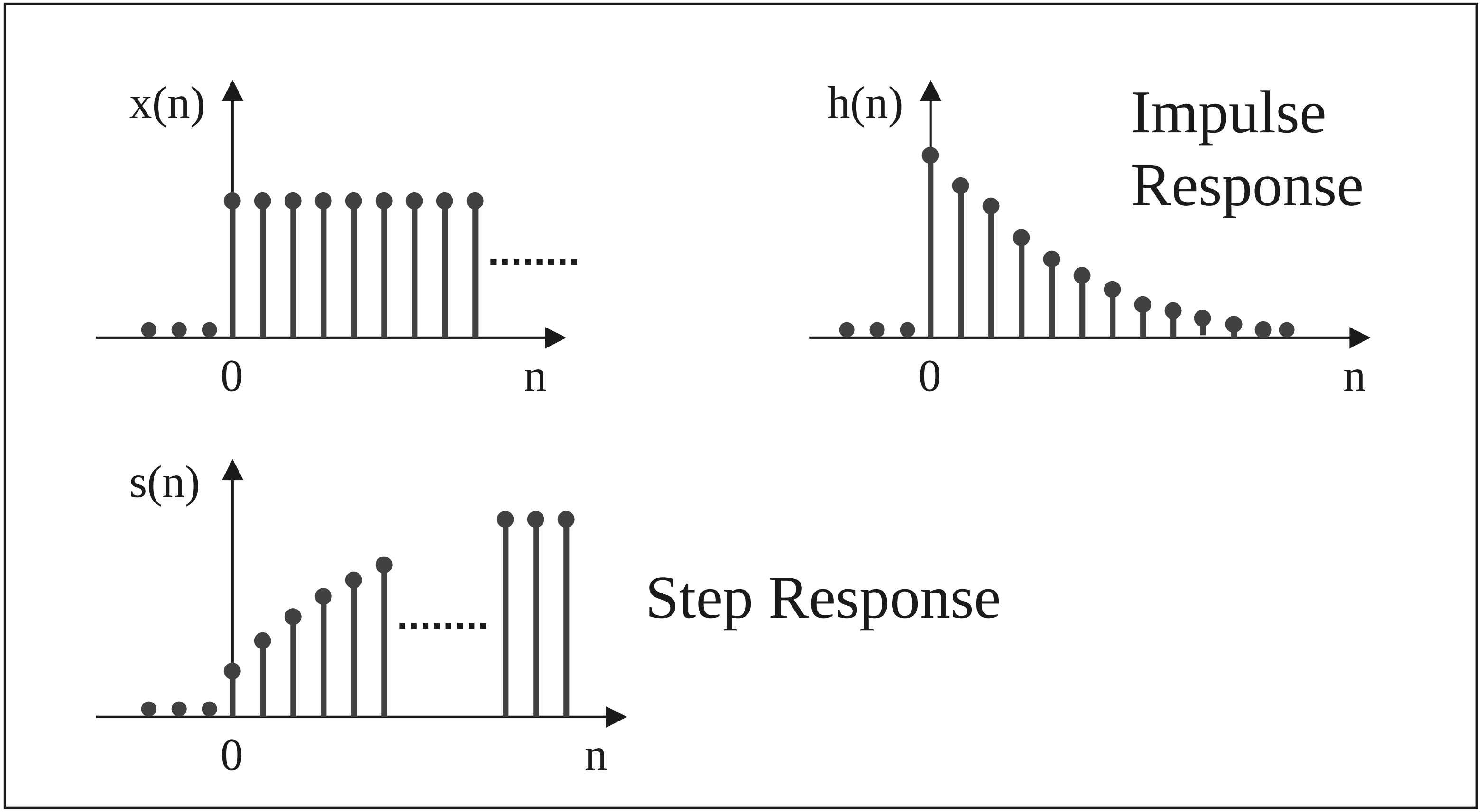
$h(2) = 0.7^2 = 0.49$ $h(3) = 0.7^3 = 0.343$

Step response is running sum of h(n) hence:

$s(0) = h(0) = 1$ $s(1) = h(0) + h(1) = 1.7$

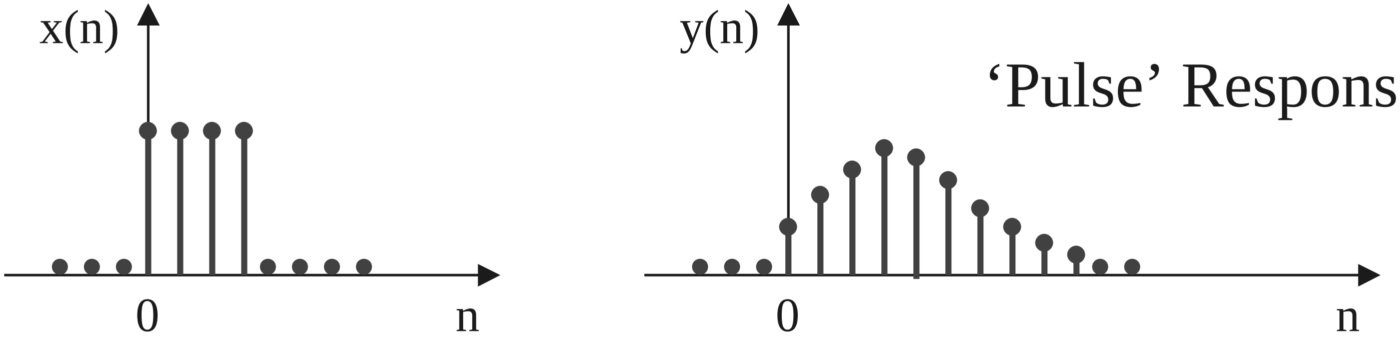
$s(2) = s(1) + h(2) = 2.19$ $s(3) = s(2) + h(3) = 2.533$





Impulse Response

Step Response



‘Pulse’ Response

- **Analysis of Systems**

  Response to Non-trivial input signals

  Procedure:

1. Decompose input signal into a sum of elementary signals
2. Find response of the system to each of the elementary signals.
3. Using linearity, determine the system response to the overall signal by summing the scaled elementary responses.

Step 1.

Elementary signals are impulses $\delta(n-k)$, $-\infty < k < \infty$

For all k, $x(n)\delta(n-k) = x(k)\delta(n-k)$ so overall input signal can be written:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

e.g. Input sequence $x(n) = \{2, 4, 0, 3\}$

$$\uparrow$$

$$x(n) = 2\,\delta(n+1) + 4\,\delta(n) + 3\,\delta(n-2)$$

Step 2

$h(n) = H [\delta(n)]$

H is time-invariant so it follows:

$h(n-k) = H [\delta(n-k)]$, for $- \infty < k < \infty$

Step 3

$y(n) = H[x(n)]$

$$y(n) = H\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

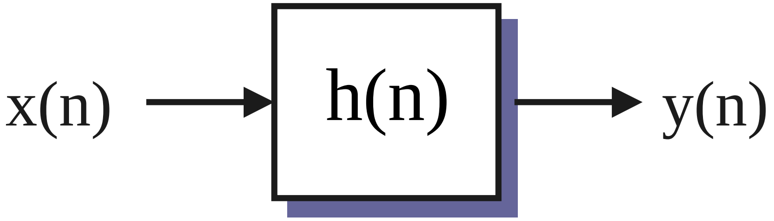
$$= \sum_{k=-\infty}^{\infty} x(k)\,H[\delta(n-k)]$$

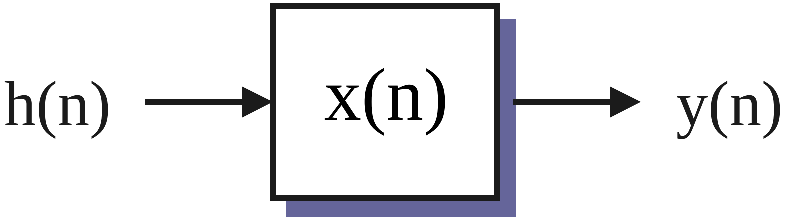
$$= \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Convolutional Sum: $y(n) = x(n) * h(n)$

1. Time reverse sequence $h(k)$: $\rightarrow h(-k)$
2. Shift (delay) sequence by n: $\rightarrow h(n-k)$
3. Multiply sequences: $x(k)h(n-k)$ for $-\infty < k < \infty$
4. Add multiplied values over all k to obtain $y(n)$
5. Repeat steps 1 to 4 for each n for which the sequences overlap.

Note: $x(n) * h(n) = h(n) * x(n)$: Commutative





● Example: Convolution

x(k) = {0, 0, 2, 1, 0….} h(k) = {0, 0, 1, 2, 3, 0….}

↑ ↑

h(-k) = {3, 2, 1, 0 ,0…}

↑

y(-2-k) = {0, 0, 2, 0, 0} = 2

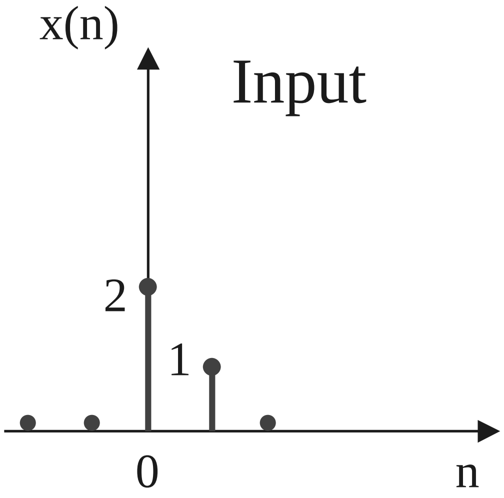
y(-1-k) = {0, 0, 4, 1, 0} = 4 + 1 = 5

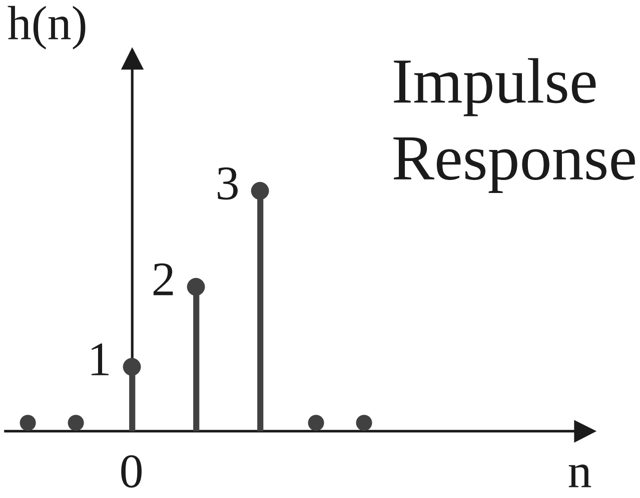
y(0-k) = {0, 0, 6, 2, 0} = 6 + 2 = 8

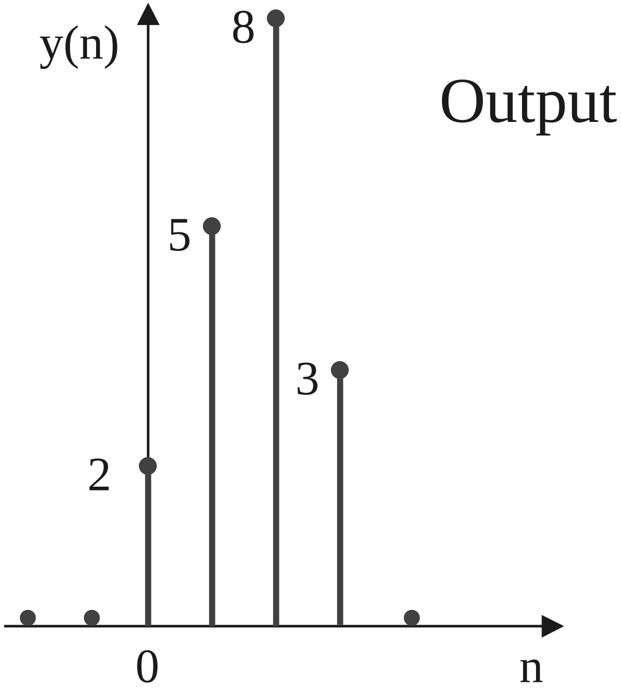
y(1-k) = {0, 0, 0, 3, 0} = 3

y(n) = {..0, 0, 2, 5, 8, 3…..}

↑


x(n)
Input
2
1
0
n


h(n)
Impulse Response
3
2
1
0
n

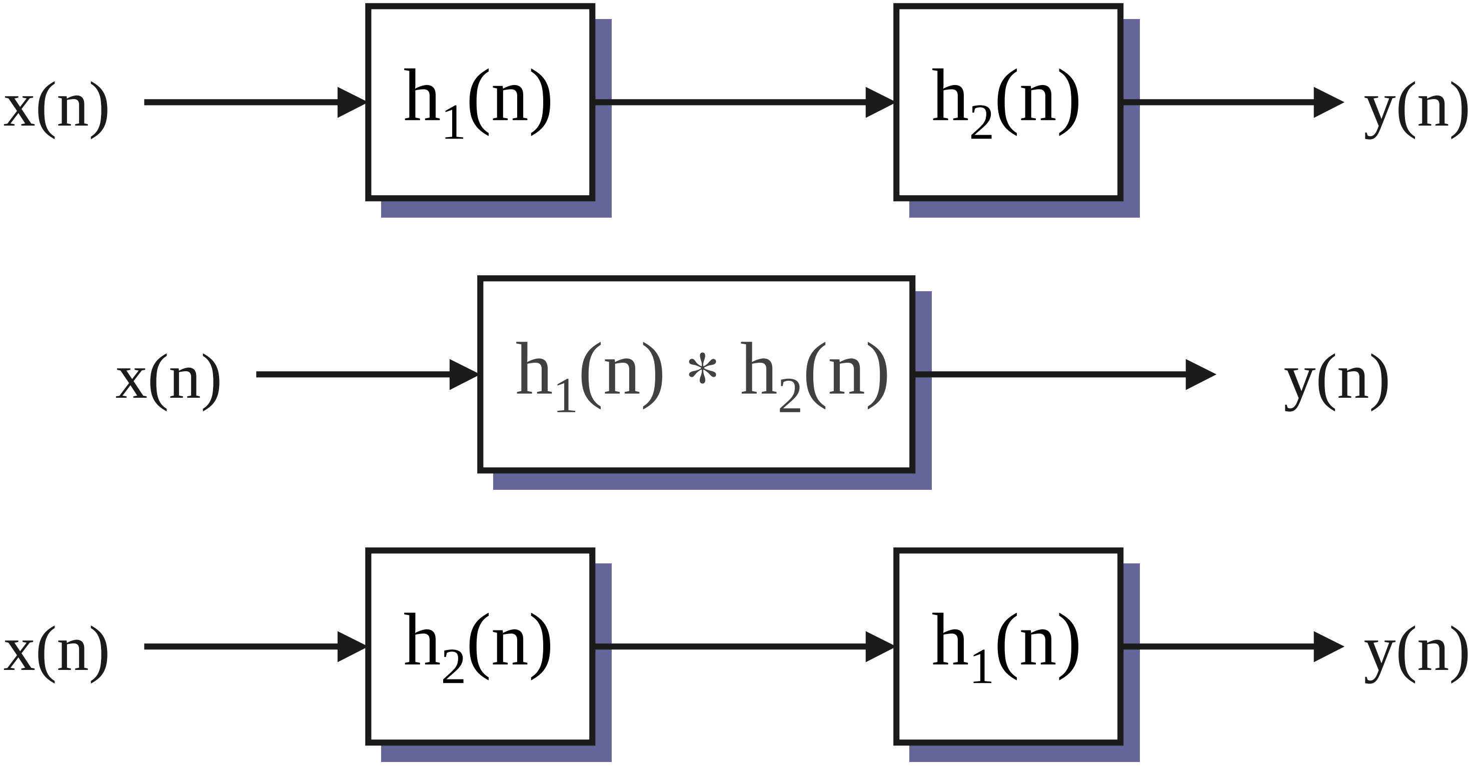

y(n)
8
Output
5
3
2
0
n

- **More properties of Convolution**

Associative

$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

Hence:

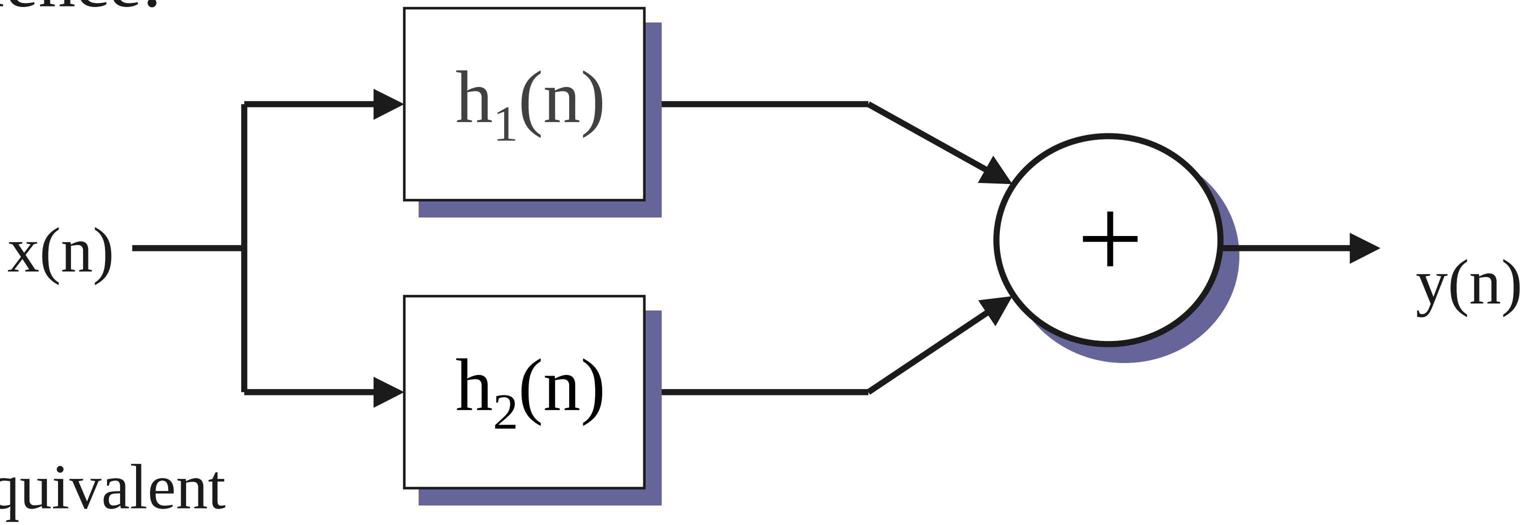


Distributive

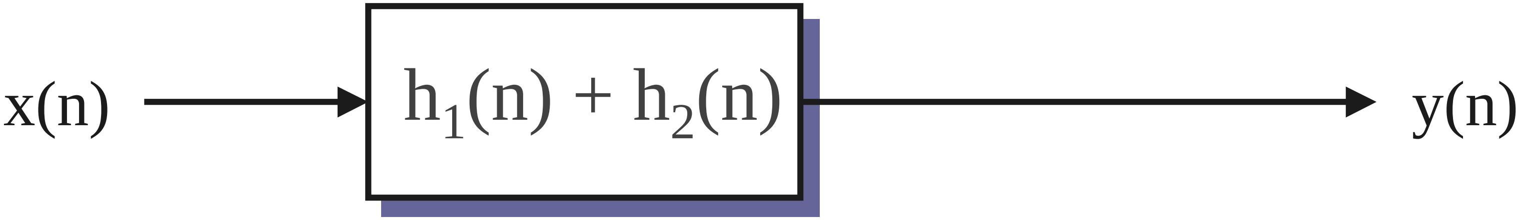
$$x(n) * [h_1(n)] + h_2(n)]$$

$$= [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

Hence:



is equivalent to:



- **Stability**

A system is said to be *stable*, if for any bounded input signal, the output is also bounded.

A *bounded* signal is one whose magnitude does not exceed a finite value. So if x(n) and y(n) are both bounded:

$$\left|x(n)\right| \leq M_X < \infty \qquad \left|y(n)\right| \leq M_Y < \infty$$

where M_X and M_Y are respective signal bounds.

Convolution can now be applied to determine a condition on the impulse response which guarantees a stable system.

$$\left|y(n)\right| = \left|\sum_{k=0}^{\infty} h(k)\, x(n-k)\right|$$

$$\leq \sum_{k=0}^{\infty} \left|h(k)\right| \left|x(n-k)\right|$$

$$\leq M_X \sum_{k=0}^{\infty} \left|h(k)\right|$$

Thus y(n) will only be bounded if the absolute values of the impulse response are *summable*. That is if:

$$\sum_{k=0}^{\infty} \left|h(k)\right| \equiv S_h < \infty$$

- **Cross-Correlation**

  Measures the similarity between two sampled signals.

  Can be used to locate a particular sequence or pattern in a noisy signal.

  Computation similar to convolution but without the time reversal.

$$r_{xy}(l) = \sum_{k=-\infty}^{\infty} x(k)\, y(k-l)$$

where l denotes time lag.

- **Computation of Cross-Correlation**:

1. Shift (delay) sequence by l: → $y(k-l)$
2. Multiply sequences: → $x(k)\, y(k-l) \quad -\infty < k < \infty$
3. Add multiplied values over all k: → $r_{xy}(l)$
4. Repeat 1 to 3 for each l.

Properties

Cross correlation written in terms of Convolution:

$$r_{xy}(l) = x(l) * y(-l)$$

Cross-correlation is not commutative, however:

$$r_{yx}(l) = y(l) * x(-l) = x(-l) * y(l) = r_{xy}(-l)$$

i.e . One is the time reverse of the other.

Example: Cross-Correlation

x(k) = {..0,0,1,2,2,1..}

↑

y(k) = {..0,1,1,3,3,1,0..}

↑

$r_{xy}(-4)$ = {..0,0,0,0,1,0,0,0,0..}   = 1

$r_{xy}(-3)$ = {..0,0,0,0,3,2,0,0,0..}   = 3 + 2 = 5

$r_{xy}(-2)$ = {..0,0,0,0,3,6,2,0,0..}   = 3 + 6 + 2 = 11

$r_{xy}(-1)$ = {..0,0,0,0,1,6,6,1,0..}   = 1+ 6 + 6 +1 = 14

$r_{xy}(0)$ = {..0,0,0,0,1,2,6,3,0..}   = 1 + 2 + 6 +3 = 12

$r_{xy}(1)$ = {..0,0,0,0,0,2,2,3,0..}   = 2 + 2 + 3 = 7

$r_{xy}(2)$ = {..0,0,0,0,0,0,2,1,0..}   = 2 + 1 = 3

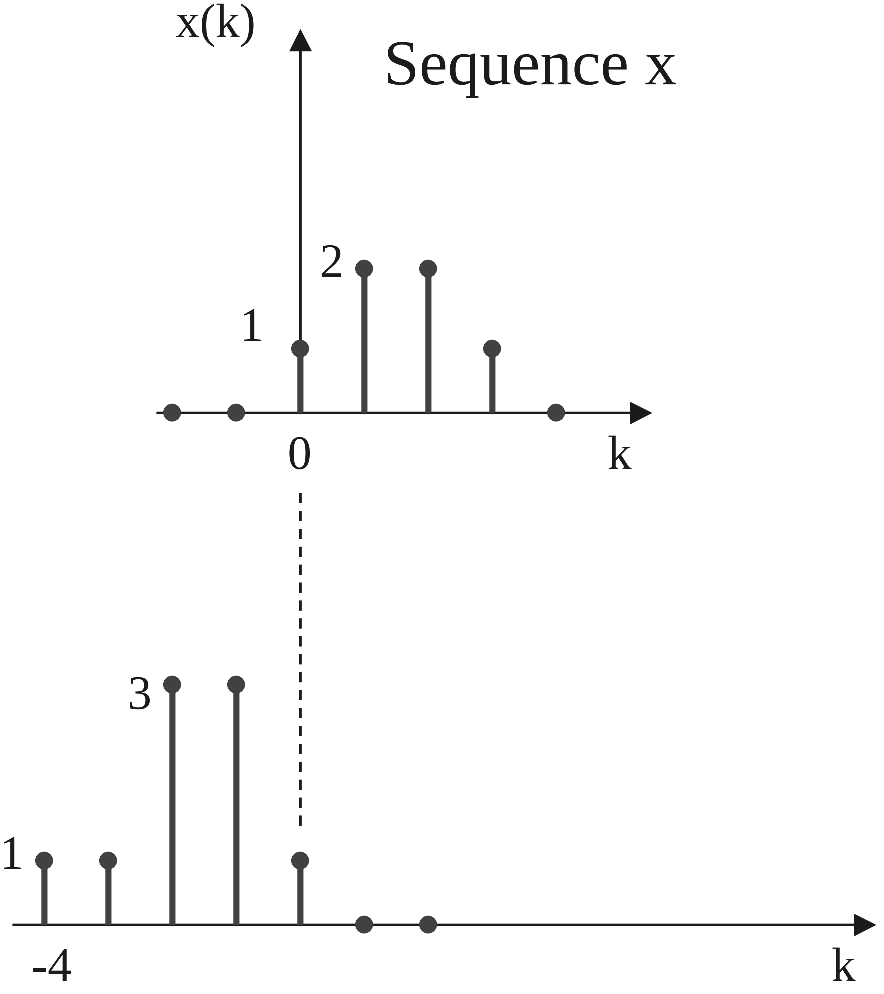
$r_{xy}(3)$ = {..0,0,0,0,0,0,0,1,0..}   = 1

Hence $r_{xy}(l)$ = {..0,1,5,11,14,12,7,3,1,0..}

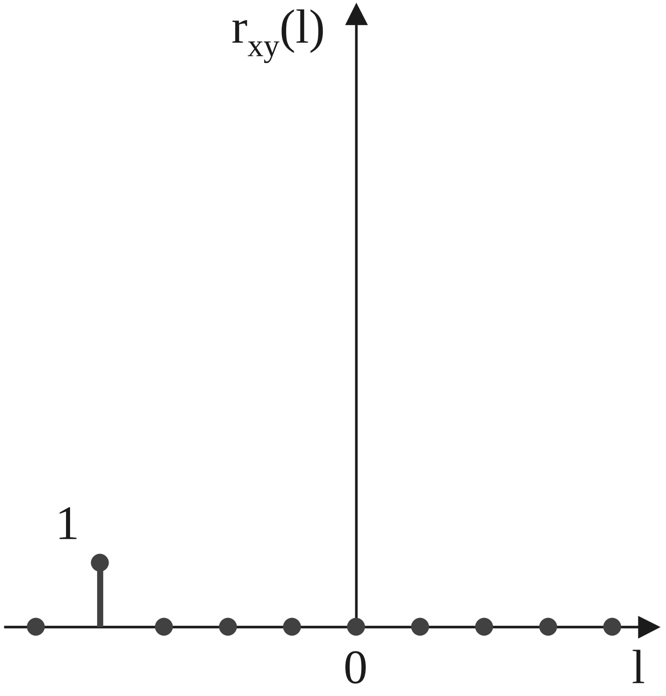
Indicates most similarity at time lag of 1.

Step 1

Sequence x

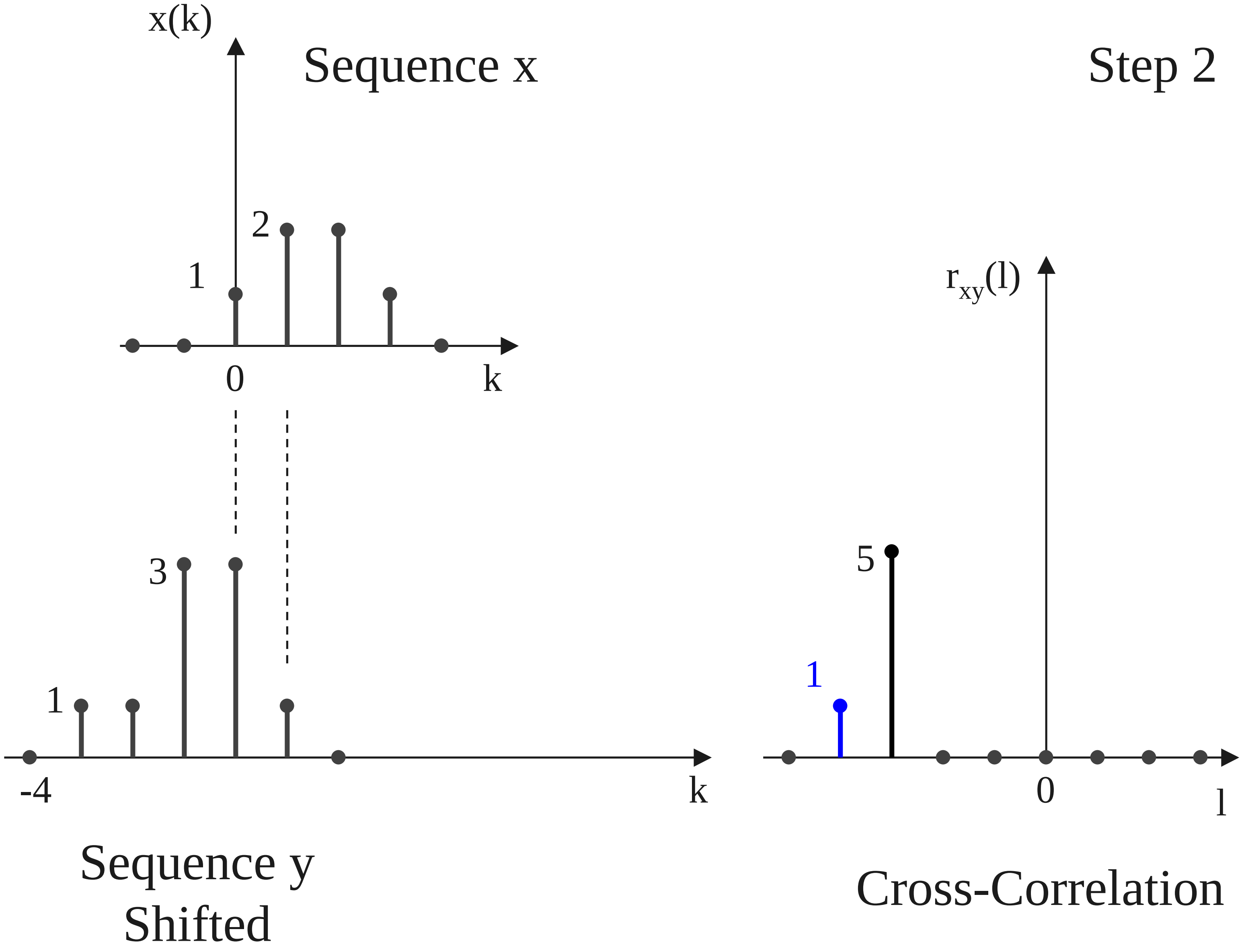


Sequence y
Shifted



Cross-Correlation

Step 2

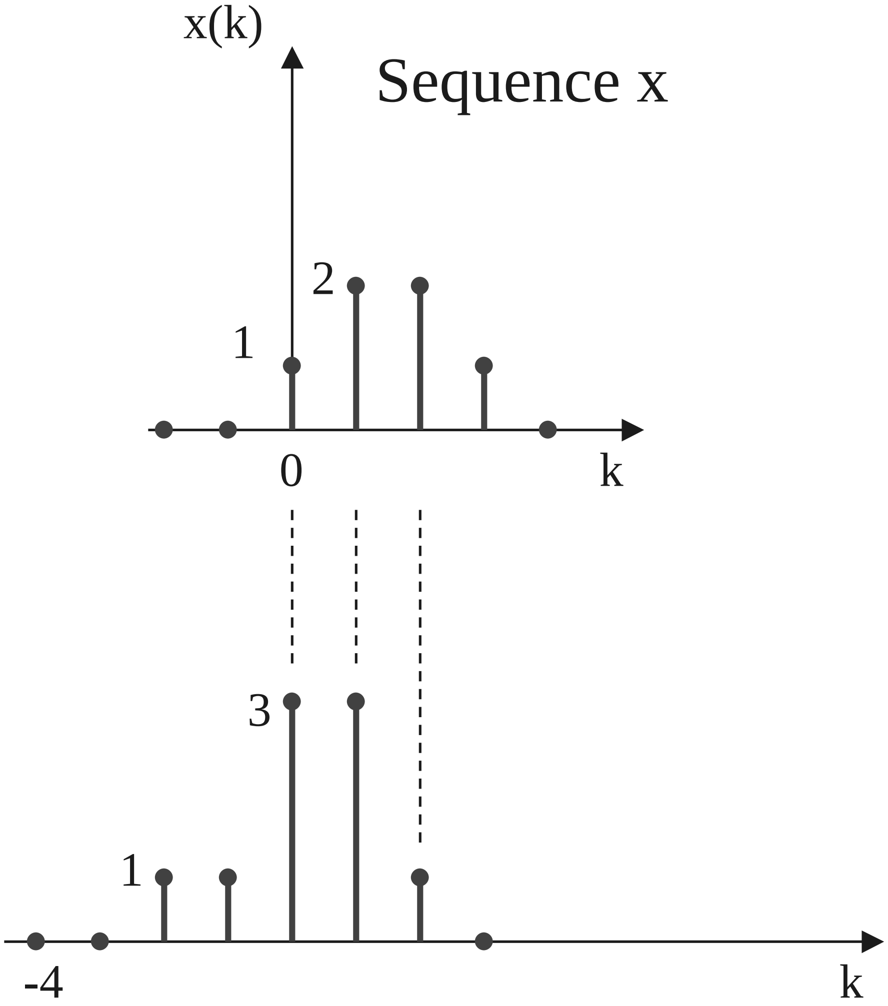


Sequence x

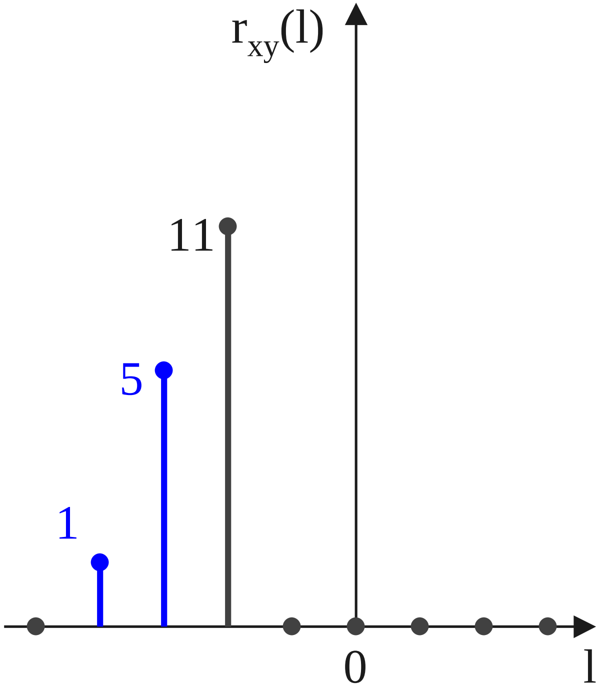
Sequence y
Shifted

Cross-Correlation

Step 3

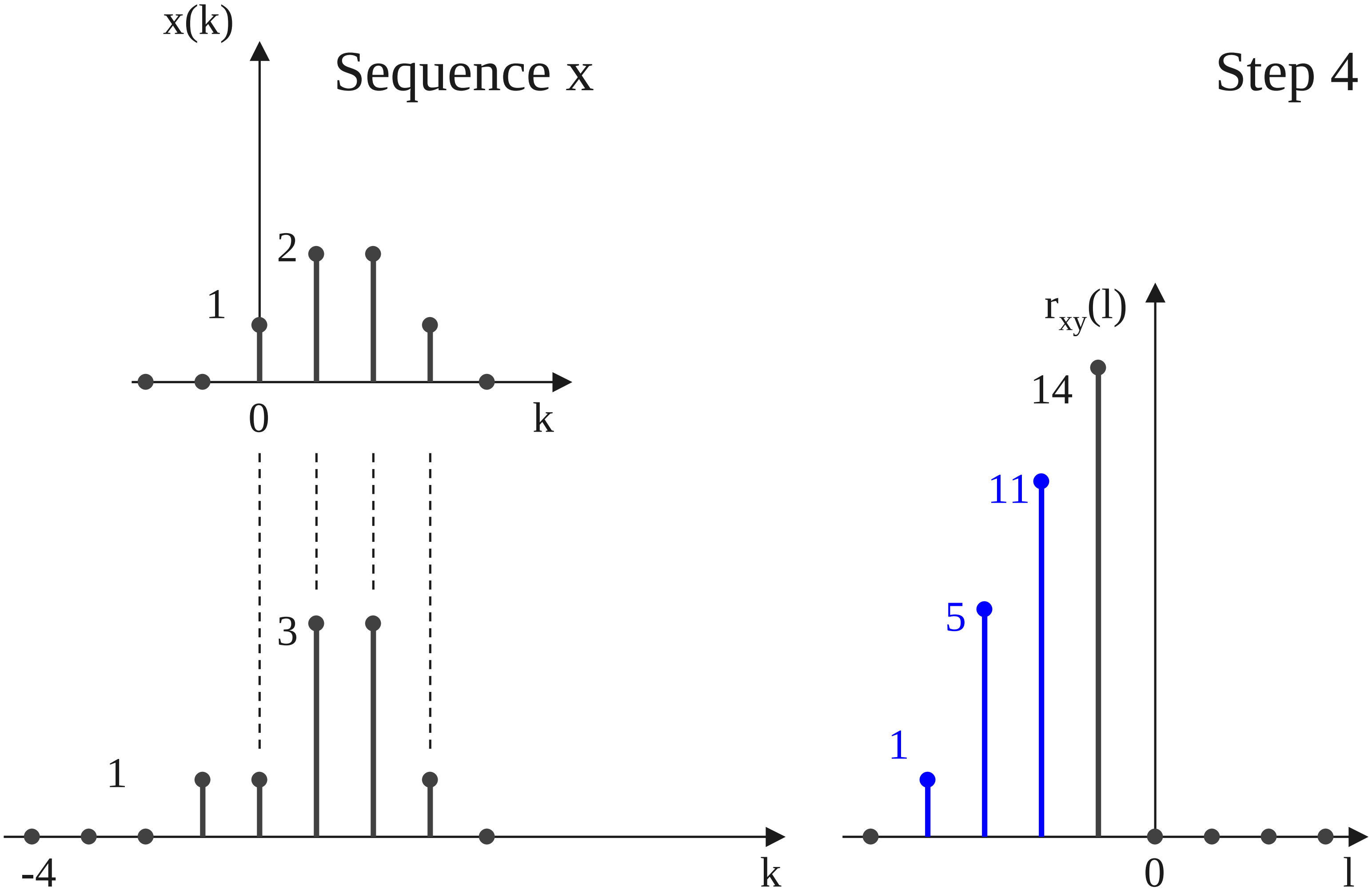


Sequence y
Shifted



Cross-Correlation

Step 4

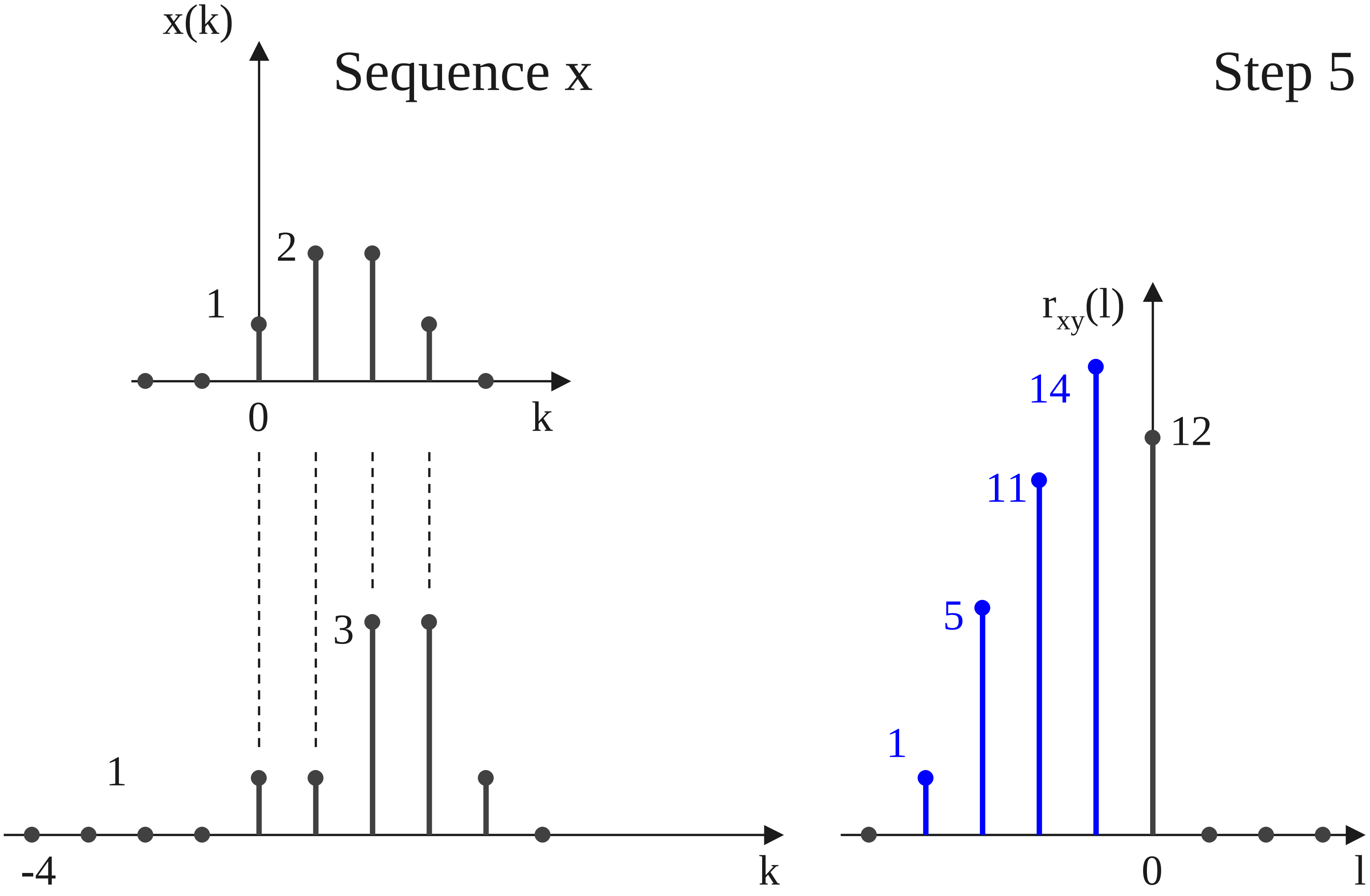


Sequence x

Sequence y
Shifted

Cross-Correlation
** Best Match **

Step 5

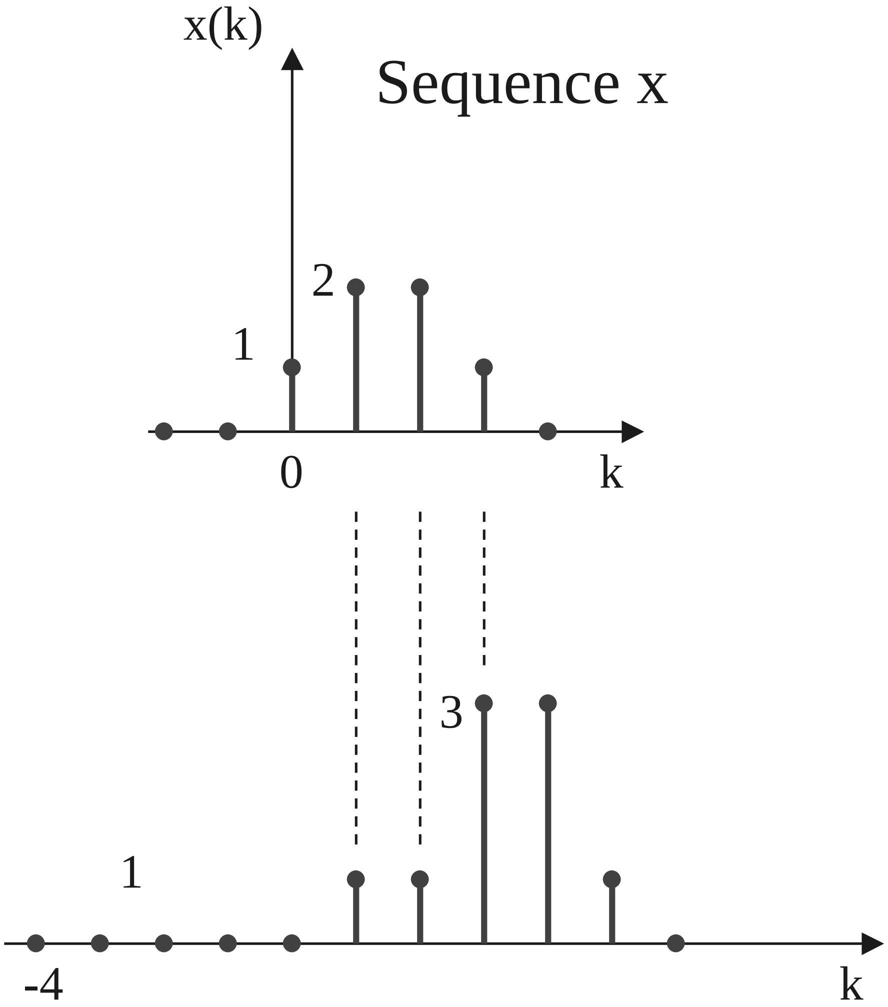


Sequence x

Sequence y
Shifted

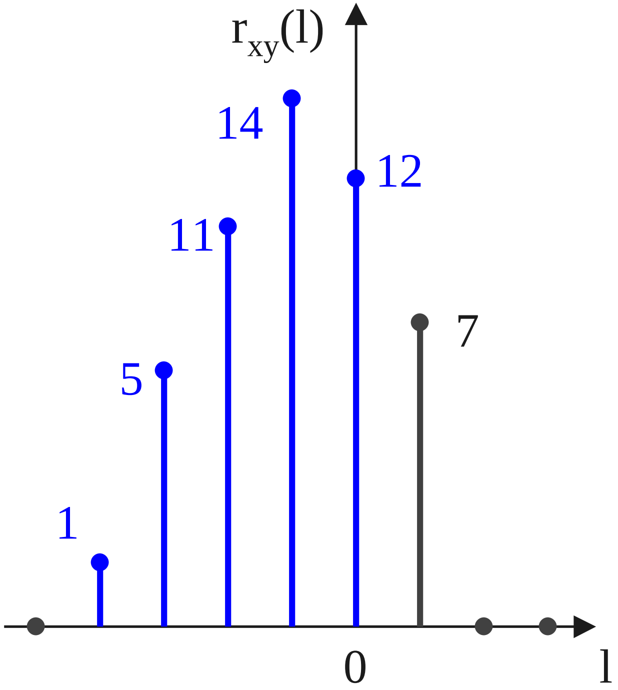
Cross-Correlation

Step 6



Sequence x

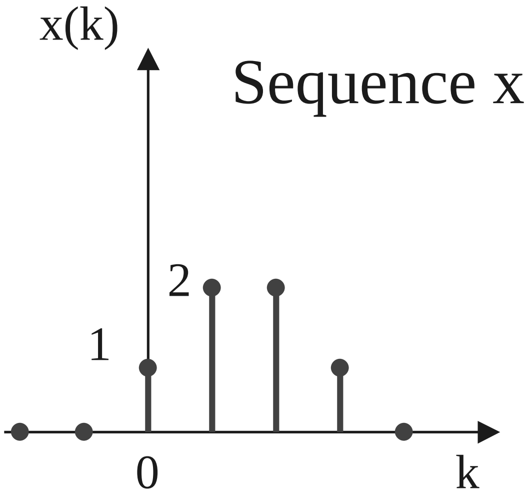
Sequence y
Shifted

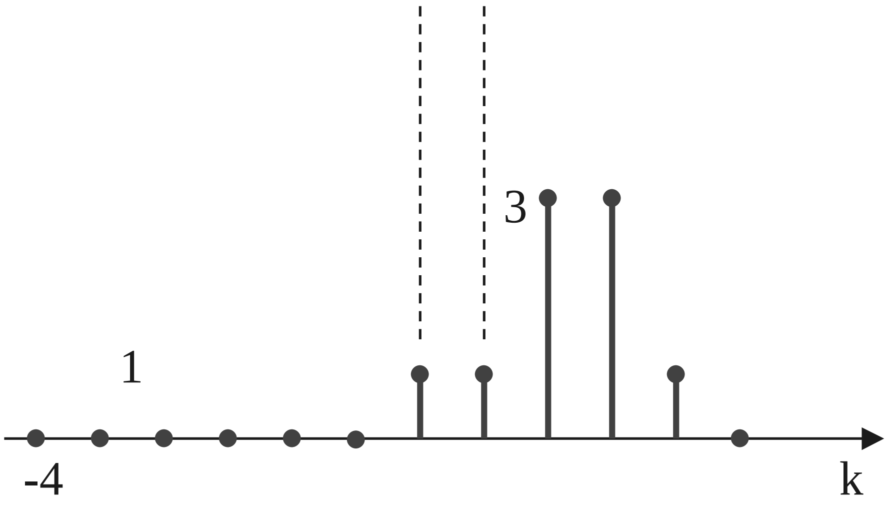


Cross-Correlation

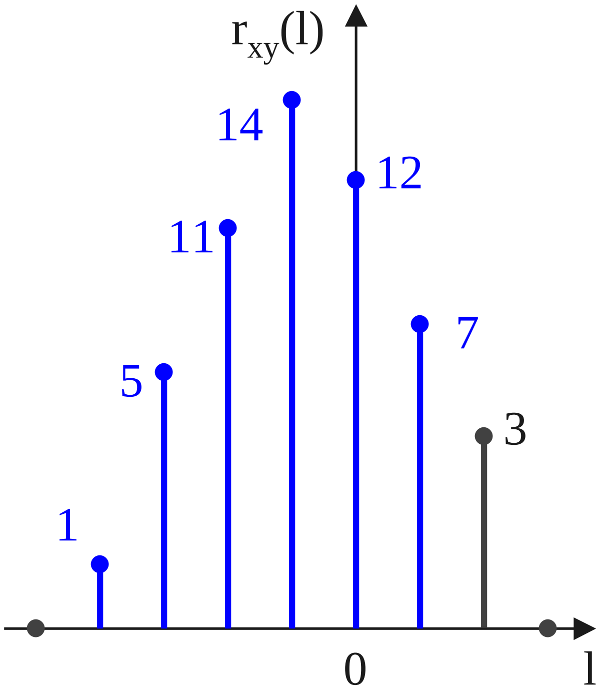
Step 7

Sequence x



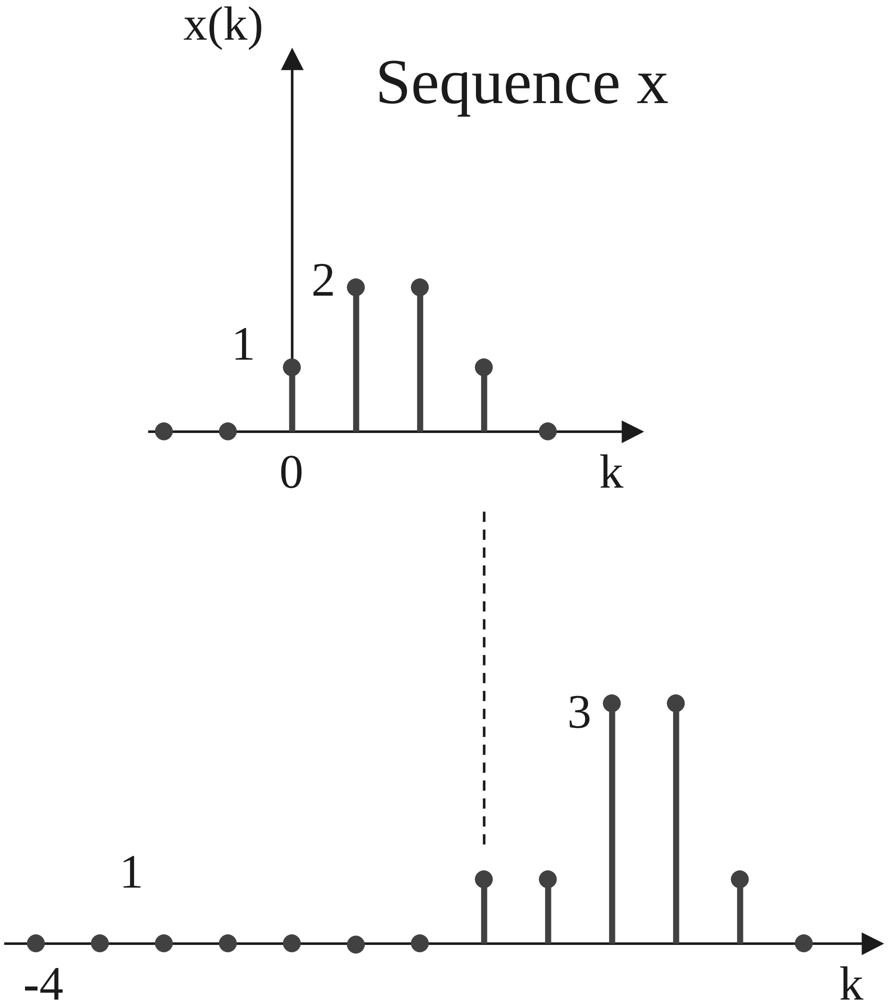


Sequence y
Shifted



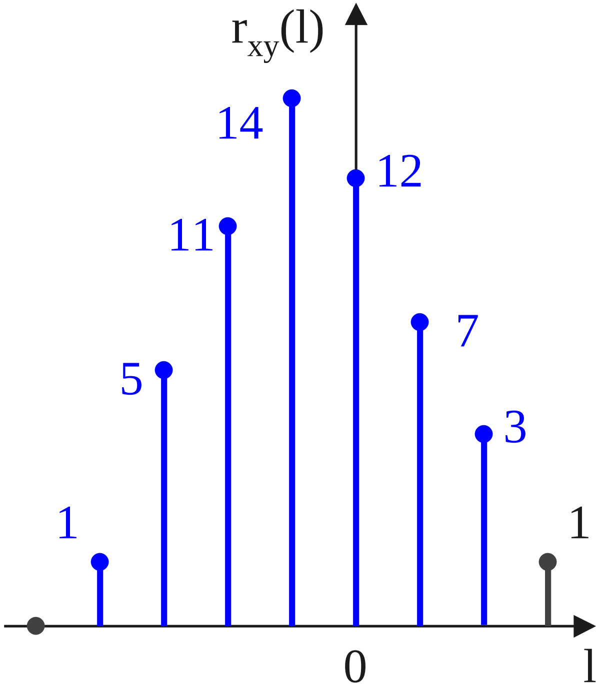
Cross-Correlation

Step 8



Sequence x

Sequence y
Shifted



Cross-Correlation

- **Auto-Correlation**

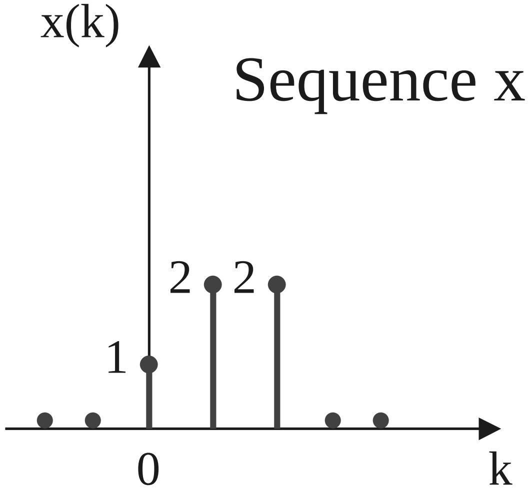
  Measures the ‘self-similarity’ of a signal with itself at different time lags.

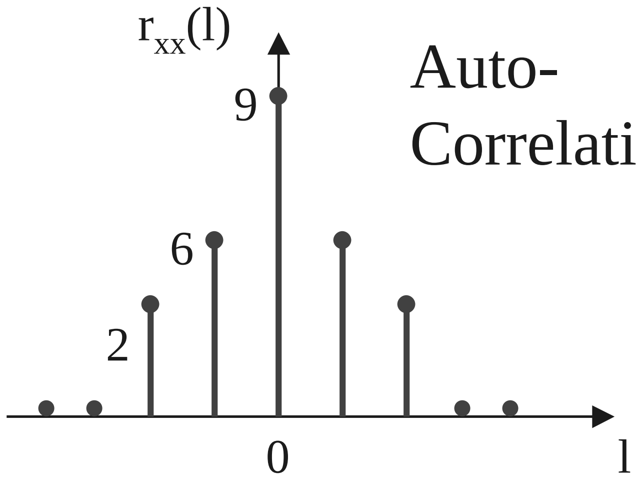
  Can be used to extract hidden periodic components in a ‘random’ signal.

$$r_{xx}(l) = \sum_{k=-\infty}^{\infty} x(k)\, x(k-l)$$

Example:

$x(k) = \{..0,0,\underset{\uparrow}{1},2,2,0..\}$ $r_{xx}(l) = \{..0,2,6,\underset{\uparrow}{9},6,2,0\ldots\}$


Sequence x


Auto-Correlation

Properties

Auto-correlation is an Even function of time around $r_{xx}(0)$.

$r_{xx}(l) = r_{xx}(-l)$ and $r_{xx}(l) <= r_{xx}(0)$ for all l

The Power of a signal is given by its auto-correlation at time lag 0:

$$r_{xx}(0) = \sum_{k=-\infty}^{\infty} [x(k)]^2$$

- **Difference Equations**

The convolutional sum may be expressed in closed-loop form rather than the sum of its inputs.

e.g. the averager:

$$y(n) = \sum_{k=0}^{N} x(n - k)$$

$$y(n-1) = \sum_{k=0}^{N} x(n-1-k)$$

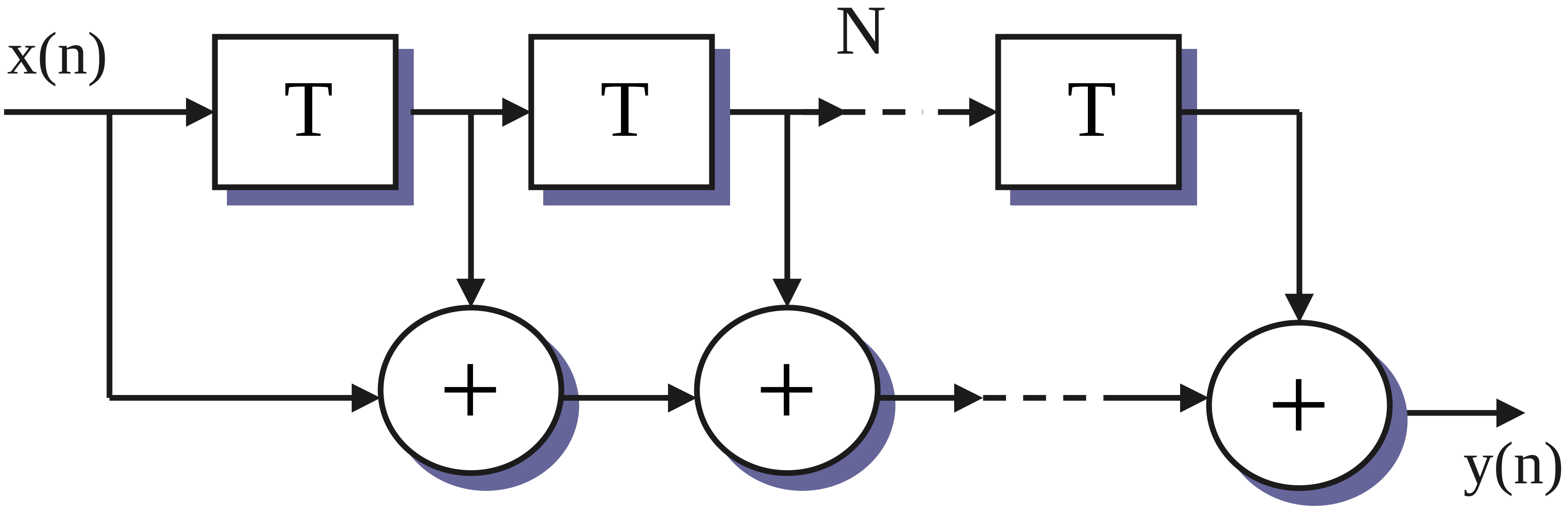
y(n) and y(n-1) have N-2 terms in common and so are related by:

$$y(n) = y(n -1) + x(n) - x(n - (N+1))$$

This is a Difference Equation.

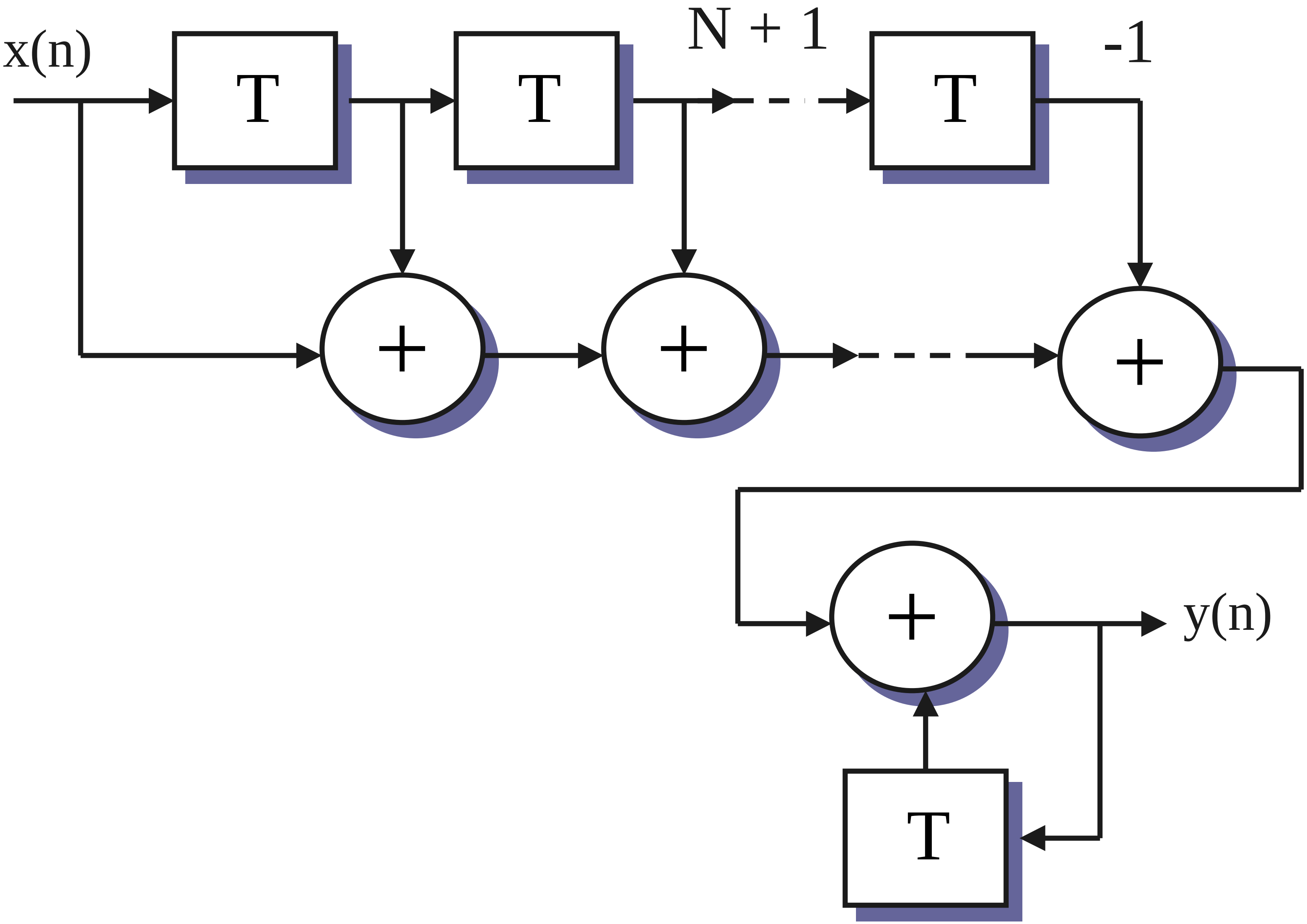
- **Realisation of Difference Equations**

Direct realisation of averager:



where T = Sampling interval

Modified form:



- **General Expression for LTI system**

$$y(n) = \sum_{k=0}^{M} a_k x(n - k) + \sum_{k=1}^{N} b_k y(n - k)$$

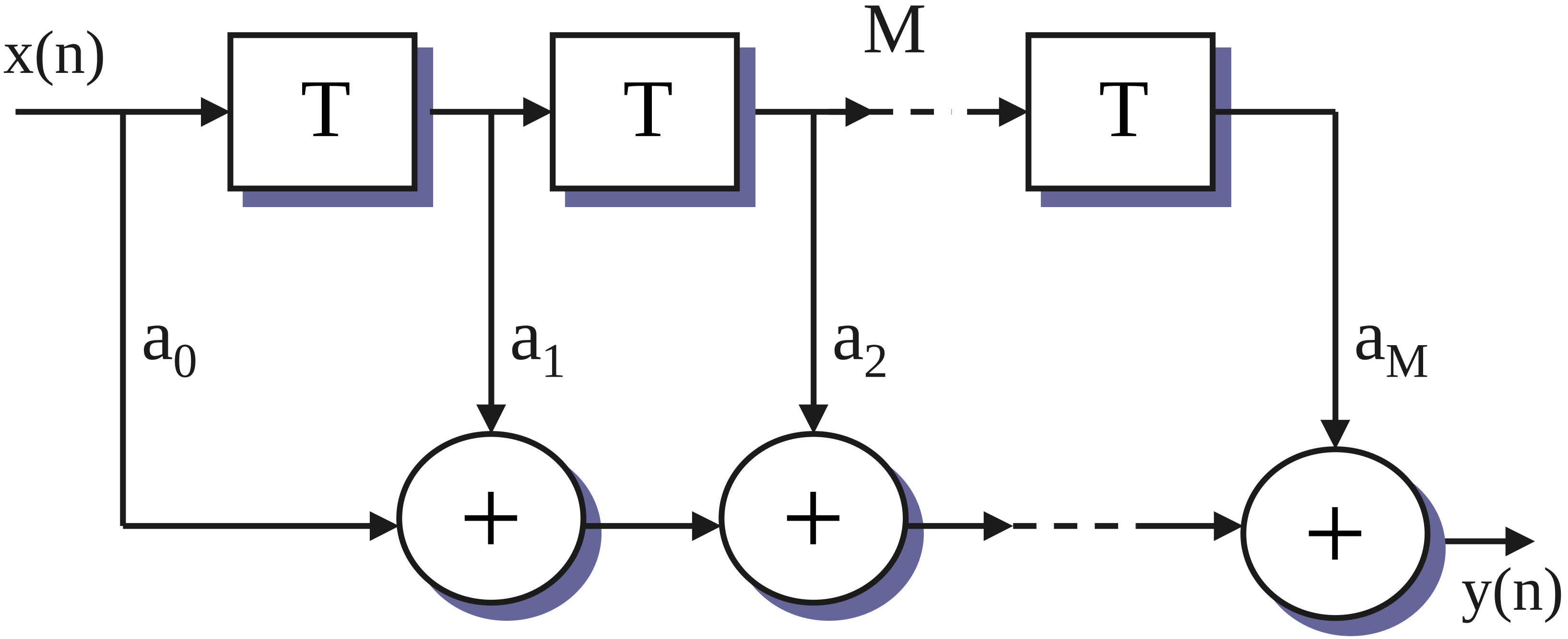
If $b_k = 0$ for all k then the system is said to be of Finite Impulse Response (FIR) provided $M < \infty$ and $a < \infty$ for all k. In other words it is FIR if the output is not fed back. An FIR system is always stable because $\sum a_k$ is always bounded.

If $b_k \neq 0$ for any k, then the system is Infinite Impulse Response and is *not necessarily* stable.

Realisation of LTI Systems

FIR case:

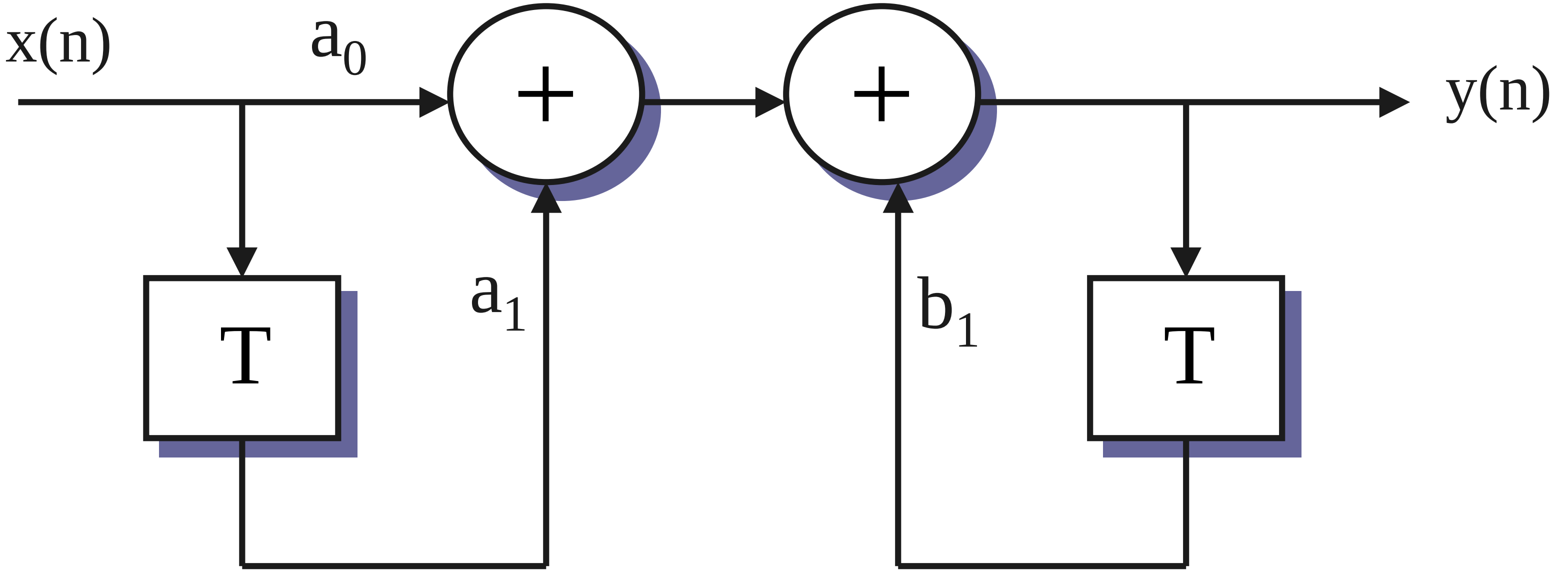
$$y(n) = \sum_{k=0}^{M} a_k x(n-k)$$

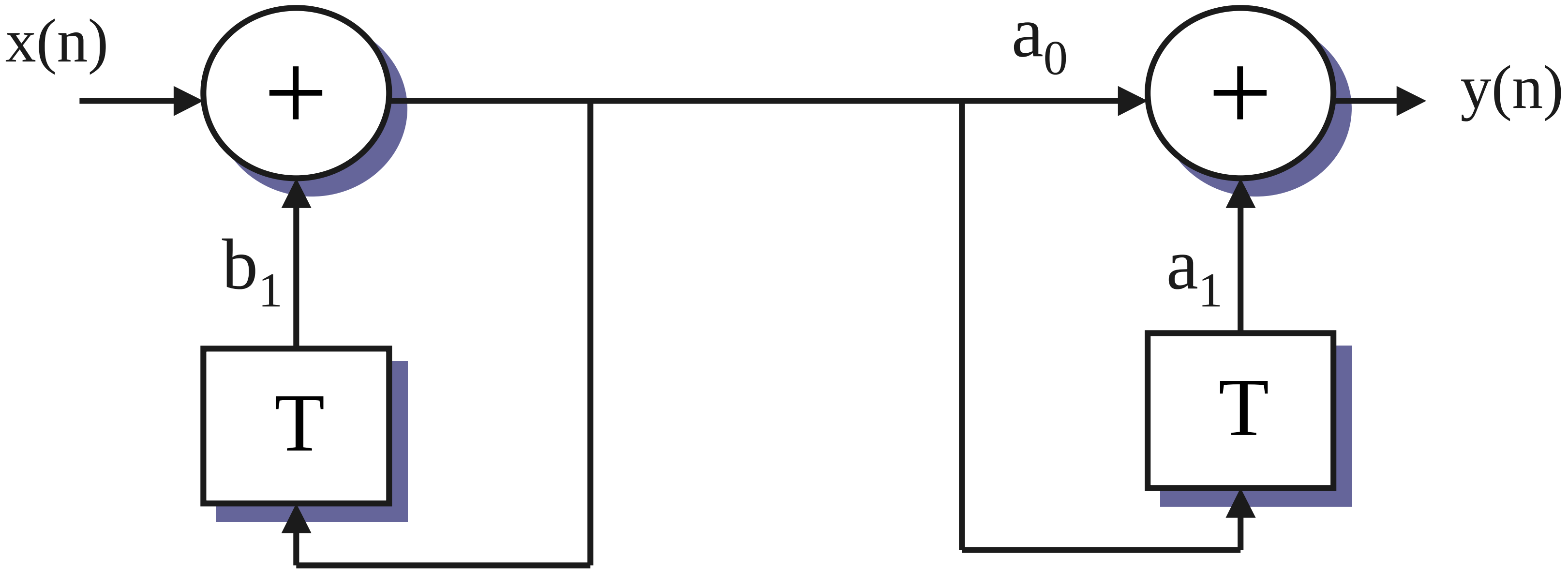


IIR Case: First Order system, M = N = 1

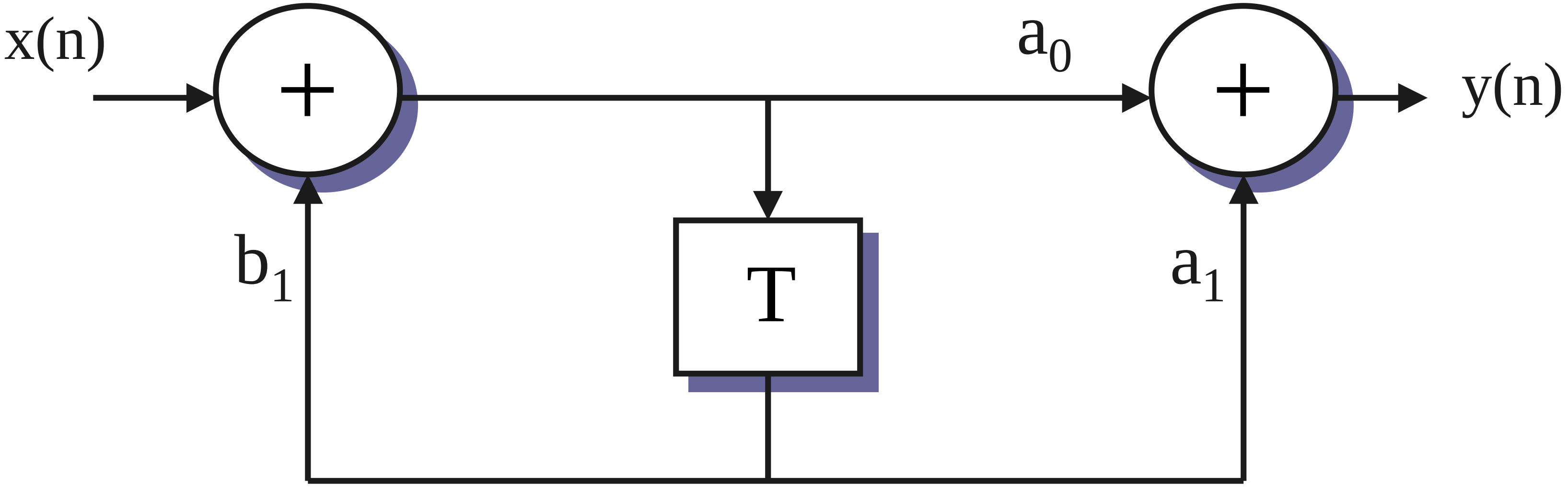
$$y(n) = a_0 x(n) + a_1 x(n-1) + b_1 y(n-1)$$

The following realisations are equivalent:





Canonical Form:



- **Impulse Response from Difference Equations**

FIR case:

$$y(n) = h(n) * x(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=0}^{M} a_k x(n-k)$$

Hence Impulse Response consists of filter coefficients a_k where k = 0,1….M:

$$h(n) = \begin{cases} a_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

e.g y(n) = 3x(n) + x(n-1) + 2x(n-2) + x(n-4)

so h(n) = {….3, 1, 2, 0, 1, 0…..}

- **Example: FIR Filter Design**

1. Define filter characteristic in the sampled frequency domain.
2. Use Inverse FFT (Fast Fourier Transform) to find filter characteristic in the time domain (Sampled Impulse response).
3. Apply ‘Window’ to remove furthest elements of Impulse Response.
4. Use suitably scaled Impulse Response values as coefficients a_0 to a_M. These are also called ‘Tap Gains’.

- **The Z-Transform**

The Fourier Transform of a discrete signal contains an infinite number of harmonics. Hence inconvenient for analysing discrete-time systems.

Replace frequency ω with complex variable z.

Definition of z-transform of discrete signal:

$$x(n) \leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}$$

e.g. if $x(n) = \{\ldots 0, \underset{\uparrow}{3}, -2, 1, -1, 0 \ldots\ldots\}$ then

$$X(z) = 3 - 2z^{-1} + z^{-2} - z^{-3}$$

Effect of Delay….

A delay of 1 sample in the time domain corresponds to multiplication by z^{-1} in the z-domain.

e.g from the previous example:

$x(n-1) = \{...0, 3, -2, 1, -1, 0.....\}$ then

↑

$x(n-1) \leftrightarrow 3z^{-1} - 2z^{-2} + z^{-3} - z^{-4} = z^{-1}X(z)$

- **Convolution**

Convolution also applies to z-transforms…

if $x(n) \leftrightarrow X(z)$ and $h(n) \leftrightarrow H(z)$ then:

$$x(n) * h(n) \leftrightarrow X(z)H(z)$$

Thus if an LTI system is defined by:

$y(n) = x(n) * h(n)$ and $y(n) \leftrightarrow Y(z)$ then:

$Y(z) = X(z)H(z)$ and $H(z) = Y(z)/X(z)$

where H(x) is defined as system's z-transfer function.

Transfer functions from Difference Equations

Transfer functions for IIR systems can be found…

e.g. LTI system $y(n) = x(n) + ay(n-1)$

Take z-transforms: $Y(z) = X(z) + az^{-1}Y(z)$

Rearranging: $Y(z)(1 - az^{-1}) = X(z)$

$$Y(z) = \frac{X(z)}{1 - az^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Rational form of H(z)

- Extension to the general case of an IIR filter

$$y(n) = \sum_{k=0}^{M} a_k x(n - k) + \sum_{k=1}^{N} b_k y(n - k)$$

$$Y(z) = X(z)\sum_{k=0}^{M} a_k z^{-k} + Y(z)\sum_{k=1}^{N} b_k z^{-k}$$

$$Y(z)\left(1 - \sum_{k=1}^{N} b_k z^{-k}\right) = X(z)\sum_{k=0}^{M} a_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 - \sum_{k=1}^{N} b_k z^{-k}}$$

- **Poles and Zeros**

A discrete system can be characterised by its poles and zeros.

Zeros are real or complex values of z for which H(z) = 0.

Poles are real or complex values of z for which H(z) = ±∞.

Any z-transfer function H(z)can be expressed in rational form: H(z) = p(z)/q(z)

where p(z) and q(z) are both polynomials in z.

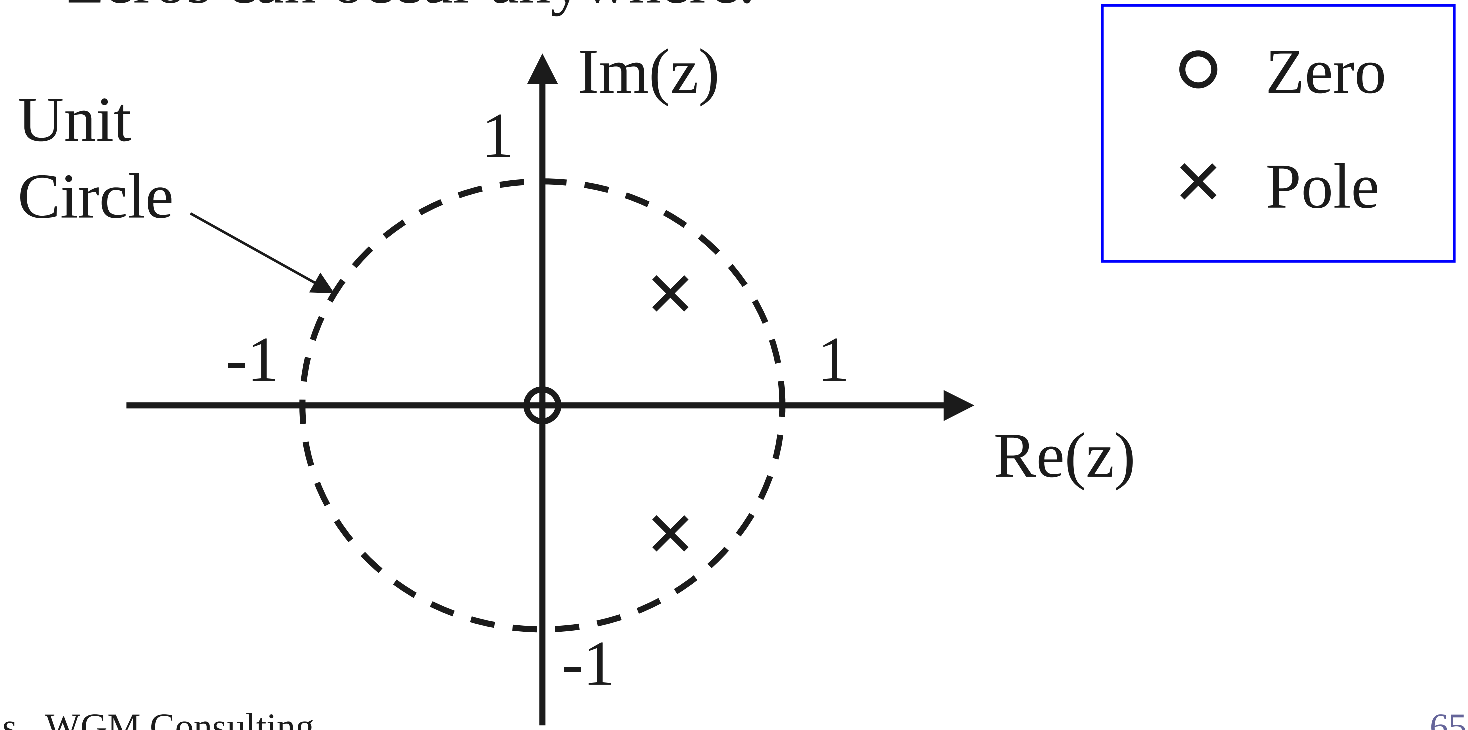
The zeros and poles of the system are the roots of p(z) and q(z) respectively.

Properties of Poles and Zeros

1 Poles and zeros are either real numbers or complex conjugate pairs (a ± jb).

2. Poles must lie inside the z-plane unit circle ($|z| < 1$), otherwise the system is unstable.

3. Zeros can occur anywhere.



- **Example: Finding Poles and Zeros**

An LTI system is described by the difference equation:

$y(n) = x(n) - 2x(n-1) + y(n-1) - 0.5y(n-2)$

Taking z-transforms:

$Y(z) = X(z) - 2z^{-1}X(z) + z^{-1}Y(z) - 0.5z^{-2}Y(z)$

Rearranging:

$Y(z)(1-z^{-1}+0.5z^{-2}) = X(z)(1-2z^{-1})$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-z^{-1}+0.5z^{-2}} = \frac{2z^{2}-4z}{2z^{2}-2z+1}$$

The zeros are the roots of $2z^2 -4z$

That is:

$z = 0$ and $z = 2$

The poles are the roots of $2z^2 - 2z +1$

That is:

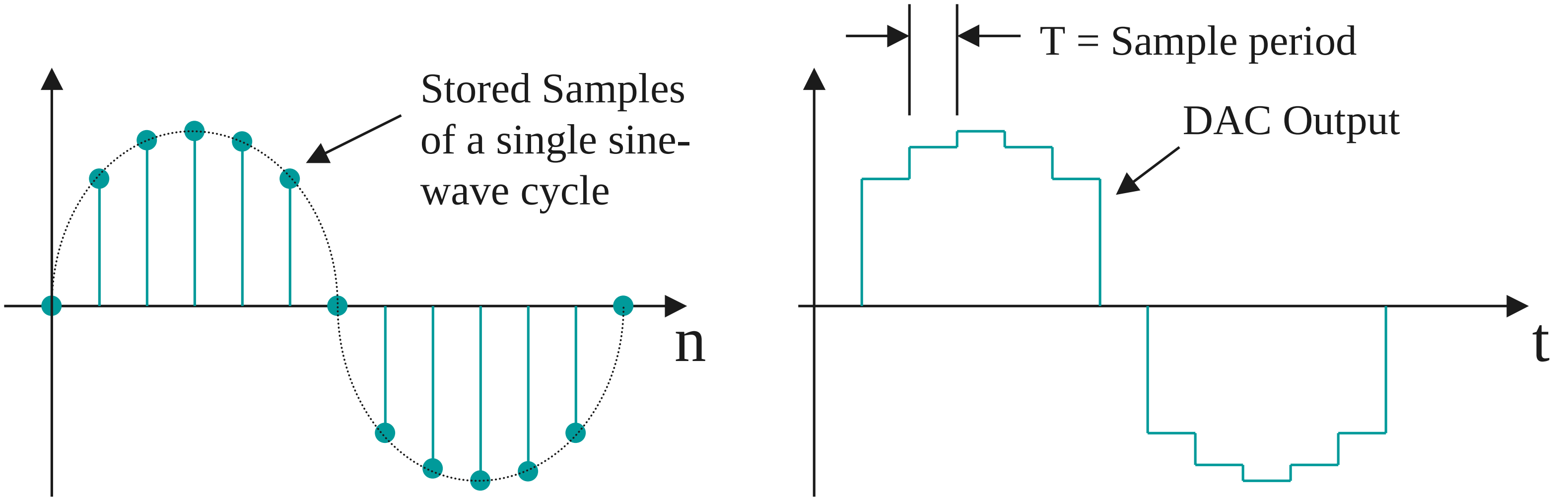
$$z = \frac{2 \pm \sqrt{-4}}{4}$$

$$z = 0.5 \pm j0.5$$

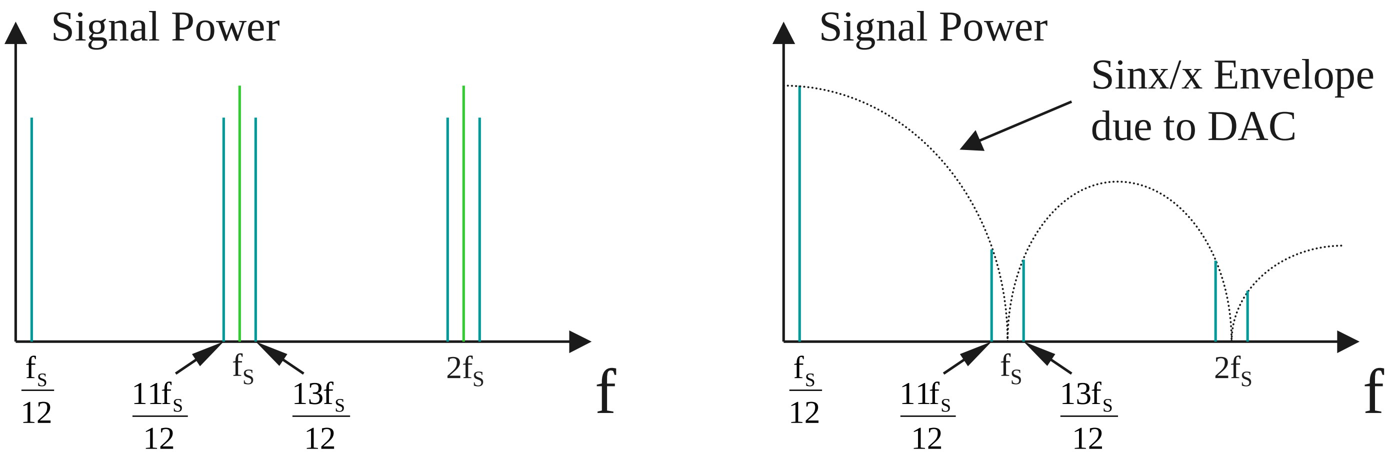
and lie inside the unit circle, since $|z| < 1$

The system is therefore *stable*.

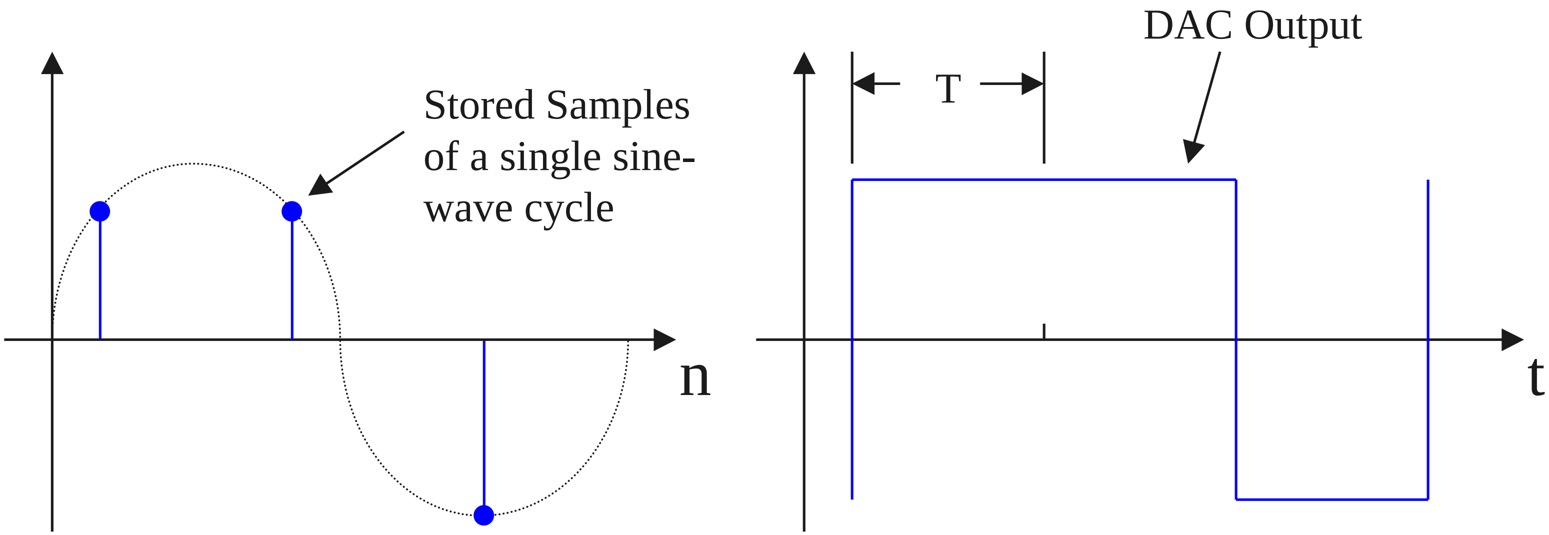
Practical Issues - Digital to Analogue



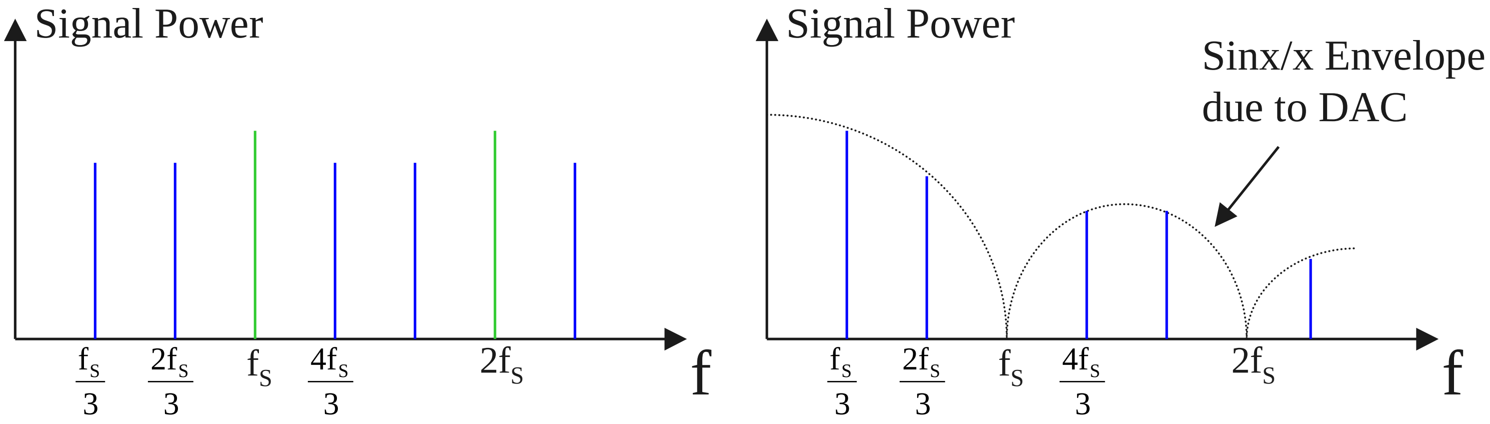
Sine wave period = 12T $f_S = 1/T$ Sine wave frequency = $f_S/12$



● Practical Issues - Lower Sampling Rate



Sine wave period = 3T $f_S = 1/T$ Sine wave frequency = $f_S/3$



● Practical Issues - Reconstruction Filter